On the Increments of the Principal Value of Brownian Local Time

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Abstract: Let $W$ be a one-dimensional Brownian motion starting from 0. Define $Y(t) = \int_0^t \frac{ds}{W(s)} := \lim_{\varepsilon \to 0} \int_0^t 1_{\{|W(s)| > \varepsilon\}} \frac{ds}{W(s)}$ as Cauchy’s principal value related to local time. We prove limsup and liminf results for the increments of $Y$.

Keywords: Brownian motion, local time, principal value, large increments.

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1. Introduction

Let \( \{W(t); t \geq 0\} \) be a one-dimensional standard Brownian motion with \( W(0) = 0 \), and let \( \{L(t, x); t \geq 0, x \in \mathbb{R}\} \) denote its jointly continuous local time process. That is, for any Borel function \( f \geq 0 \),

\[
\int_0^t f(W(s)) \, ds = \int_{-\infty}^\infty f(x) L(t, x) \, dx, \quad t \geq 0.
\]

We are interested in the process

\[(1.1) \quad Y(t) := \int_0^t \frac{ds}{W(s)}, \quad t \geq 0.\]

Rigorously speaking, the integral \( \int_0^t ds/W(s) \) should be considered in the sense of Cauchy’s principal value, i.e., \( Y(t) \) is defined by

\[(1.2) \quad Y(t) := \lim_{\varepsilon \to 0^+} \int_0^t \frac{ds}{W(s)} \mathbb{1}_{\{|W(s)| \geq \varepsilon\}} = \int_0^\infty \frac{L(t, x) - L(t, -x)}{x} \, dx.\]

Since \( x \mapsto L(t, x) \) is Hölder continuous of order \( \nu \), for any \( \nu < 1/2 \), the integral on the extreme right in (1.2) is almost surely absolutely convergent for all \( t > 0 \). The process \( \{Y(t), t \geq 0\} \) is called the principal value of Brownian local time.

It is easily seen that \( Y(\cdot) \) inherits a scaling property from Brownian motion, namely, for any fixed \( a > 0 \), \( t \mapsto a^{-1/2} Y(at) \) has the same law as \( t \mapsto Y(t) \). Although some properties distinguish \( Y(\cdot) \) from Brownian motion (in particular, \( Y(\cdot) \) is not a semimartingale), it is a kind of folklore that the asymptotic behaviors of \( Y \) are somewhat like that of a Brownian motion. For detailed studies and surveys on principal value, and relation to Hilbert transform see Biane and Yor [4], Fitzsimmons and Getoor [13], Bertoin [2], [3], Yamada [20], Boufoussi et al. [5], Ait Ouahra and Eddahbi [1], Csáki et al. [11] and a collection of papers [22] together with their references. Biane and Yor [4] presented a detailed study on \( Y \) and determined a number of distributions for principal values and related processes.

Concerning almost sure limit theorems for \( Y \) and its increments, we summarize the relevant results in the literature. It was shown in [17] that the following law of the iterated logarithm holds:

**Theorem A.** (Hu and Shi [17])

\[(1.3) \quad \limsup_{T \to \infty} \frac{Y(T)}{\sqrt{T \log \log T}} = \sqrt{8}, \quad \text{a.s.}\]

This was extended in [10] to a Strassen-type [18] functional law of the iterated logarithm.
**Theorem B.** (Csáki et al. [10]) With probability one the set

\[
\left\{ \frac{Y(xT)}{\sqrt{8T\log \log T}}, \ 0 \leq x \leq 1 \right\}_{T \geq 3}
\]

is relatively compact in $C[0, 1]$ with limit set equal to

\[
\mathcal{S} := \left\{ f \in C[0, 1] : f(0) = 0, \ f \text{ is absolutely continuous and } \int_0^1 (f'(x))^2 \, dx \leq 1 \right\}.
\]

Concerning Chung-type law of the iterated logarithm, we have the following result:

**Theorem C.** (Hu [16])

\[
\lim \inf_{T \to \infty} \sqrt{\log \log T} \sup_{0 \leq s \leq T} |Y(s)| = K_1, \quad \text{a.s.}
\]

with some (unknown) constant $K_1 > 0$.

The large increments were studied in [7] and [8]:

**Theorem D.** (Csáki et al. [7]) Under the conditions

\[
0 < a_T \leq T,
\]

\[
T \mapsto a_T \text{ and } T \mapsto T/a_T \text{ are both non-decreasing,}
\]

\[
\lim_{T \to \infty} \frac{\log(T/a_T)}{\log \log T} = \infty,
\]

we have

\[
\lim_{T \to \infty} \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} \frac{|Y(t + s) - Y(t)|}{\sqrt{a_T \log(T/a_T)}} = 2, \quad \text{a.s.}
\]

Wen [19] studied the lag increments of $Y$ and among others proved the following results.

**Theorem E.** (Wen [19])

\[
\lim \sup_{T \to \infty} \sup_{0 \leq t \leq T} \frac{\sup_{s \leq t \leq T} |Y(s) - Y(s - t)|}{\sqrt{t \log(T/t) + 2 \log \log t}} = 2, \quad \text{a.s.}
\]

Under the conditions $0 < a_T \leq T$, $a_T \to \infty$ as $T \to \infty$, we have

\[
\lim \sup_{T \to \infty} \sup_{0 \leq t \leq T - a_T} \frac{\sup_{s \leq t \leq T} |Y(t + s) - Y(t)|}{\sqrt{a_T \log((t + a_T)/a_T) + 2 \log \log a_T}} \leq 2, \quad \text{a.s.}
\]

If $a_T$ is onto, then we have equality in (1.10).
In this note our aim is to investigate further limsup and liminf behaviors of the increments of \( Y \).

**Theorem 1.1.** Assume that \( T \mapsto a_T \) is a function such that \( 0 < a_T \leq T \), and both \( a_T \) and \( T/a_T \) are non-decreasing. Then

(i)

\[
\limsup_{T \to \infty} \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} \frac{|Y(t+s) - Y(t)|}{\sqrt{a_T \left( \log \sqrt{T/a_T} + \log \log T \right)}} = 8, \quad \text{a.s.}
\]

(ii) If \( a_T > T(\log T)^{-\alpha} \) for some \( \alpha < 2 \), then

\[
\liminf_{T \to \infty} \frac{\sqrt{\log \log T}}{a_T} \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |Y(t+s) - Y(t)| = K_2, \quad \text{a.s.}
\]

(iii) If \( a_T \leq T(\log T)^{-\alpha} \) for some \( \alpha > 2 \), then

\[
\liminf_{T \to \infty} \frac{\sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |Y(t+s) - Y(t)|}{\sqrt{a_T \log(T/a_T)}} = K_3, \quad \text{a.s.}
\]

with some positive constants \( K_2, K_3 \). If, moreover,

\[
\lim_{T \to \infty} \frac{\log(T/a_T)}{\log \log T} = \infty,
\]

then \( K_3 = 2 \).

**Theorem 1.2.** Assume that \( T \mapsto a_T \) is a function such that \( 0 < a_T \leq T \), and both \( a_T \) and \( T/a_T \) are non-decreasing. Then

(i)

\[
\liminf_{T \to \infty} \frac{\sqrt{T \log \log T}}{a_T} \inf_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |Y(t+s) - Y(t)| = K_4, \quad \text{a.s.}
\]

with some positive constant \( K_4 \). If \( \lim_{T \to \infty}(a_T/T) = 0 \), then \( K_4 = 1/\sqrt{2} \).

(ii) If \( 0 < \lim_{T \to \infty}(a_T/T) = \rho \leq 1 \), then

\[
\limsup_{T \to \infty} \frac{\inf_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |Y(t+s) - Y(t)|}{\sqrt{T \log \log T}} = \rho \sqrt{8}, \quad \text{a.s.}
\]

(iii) If

\[
\lim_{T \to \infty} \frac{a_T(\log \log T)^2}{T} = 0,
\]

then

\[
\limsup_{T \to \infty} \frac{\sqrt{T}}{a_T \sqrt{\log \log T}} \inf_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |Y(t+s) - Y(t)| = K_5, \quad \text{a.s.}
\]

with some positive constant \( K_5 \).
Remark 1. The exact values of the constants $K_i$, $i = 2, 3, 4, 5$ are unknown in general and it seems difficult to determine them except in certain particular cases. In the proofs we establish different upper and lower bounds. It follows however by 0-1 law for Brownian motion that the limsup’s and liminf’s considered here are non-random constants.

Remark 2. Plainly we recover some previous results on the path properties of $Y$ by considering particular cases of Theorems 1.1 and 1.2. For instance, Theorems A and C follow from (1.11) and (1.12) respectively by taking $a_T = T$, and (1.8) follows from (1.11) combining with (1.13). However in Theorem 1.1(ii) and Theorem 1.2(ii) there are still small gaps in $a_T$.

The organization of the paper is as follows: In Section 2 some facts are presented needed in the proofs. Section 3 contains the necessary probability estimates. Theorem 1.1(i) and Theorem 1.1(iia,b) are proved in Sections 4 and 5, resp., while Theorem 1.2(i) and Theorem 1.2(iia,b) are proved in Sections 6 and 7, resp.

Throughout the paper, the letter $K$ with subscripts will denote some important but unknown finite positive constants, while the letter $c$ with subscripts denotes some finite and positive universal constants not important in our investigations. When the constants depend on a parameter, say $\delta$, they are denoted by $c(\delta)$ with subscripts.

2. Facts

Let $\{W(t), t \geq 0\}$ be a standard Brownian motion and define the following objects:

\begin{align}
(2.1) \quad g & := \sup\{t : t \leq 1, W(t) = 0\} \\
(2.2) \quad B(s) & := \frac{W(sg)}{\sqrt{g}}, \quad 0 \leq s \leq 1, \\
(2.3) \quad m(s) & := \frac{|W(g + s(1 - g))|}{\sqrt{1 - g}}, \quad 0 \leq s \leq 1.
\end{align}

Here we summarize some well-known facts needed in our proofs.

Fact 2.1. (Biane and Yor [4])

\begin{equation}
\frac{\mathbb{P}(Y(1) \in dx)}{dx} = \sqrt{\frac{2}{\pi^3}} \sum_{k=0}^{\infty} (-1)^k \exp\left(-\frac{(2k + 1)^2 x^2}{8}\right), \quad x \in \mathbb{R}.
\end{equation}

Consequently we have the estimate: for $\delta > 0$

\begin{equation}
c_1 \exp\left(-\frac{z^2}{8(1 - \delta)}\right) \leq \mathbb{P}(Y(1) \geq z) \leq \exp\left(-\frac{z^2}{8}\right), \quad z \geq 1
\end{equation}
with some positive constant \( c_1 = c_1(\delta) \). Moreover, \( g \), \( \{B(s), 0 \leq s \leq 1\} \) and \( \{m(s), 0 \leq s \leq 1\} \) are independent, \( g \) has arcsine distribution, \( B \) is a Brownian bridge and \( m \) is a Brownian meander.

\[
\mathbb{P} \left( \int_0^1 \frac{dv}{m(v)} < z \bigg| m(1) = 0 \right) = \sum_{k=-\infty}^{\infty} (1 - k^2 z^2) \exp \left(-\frac{k^2 z^2}{2}\right) = \frac{8\pi^2 \sqrt{2\pi}}{z^3} \sum_{k=1}^{\infty} \exp \left(-\frac{2k^2 \pi^2}{z^2}\right), \quad z > 0.
\]

\[
\mathbb{P}(m(1) > x) = e^{-x^2/2}, \quad x > 0.
\]

**Fact 2.2.** (Yor [21, Exercise 3.4 and pp. 44]) Let \( Q_{x \to 0}^\delta \) be the law of the square of a Bessel bridge from \( x \) to 0 of dimension \( \delta > 0 \) during time interval \([0,1]\). The process \((m^2(1-v), 0 \leq v \leq 1)\) conditioned on \( \{m^2(1) = x\} \) is distributed as \( Q_{x \to 0}^3 \). Furthermore, we have

\[
Q_{x \to 0}^\delta = Q_{x \to 0}^0 * Q_{x \to 0}^0, \quad \forall \delta > 0, \ x > 0,
\]

where \( * \) denotes convolution operator. Consequently, for any \( x > 0 \)

\[
\mathbb{P}\left( \int_0^1 \frac{dv}{m(v)} < z \bigg| m(1) = x \right) \geq \mathbb{P}\left( \int_0^1 \frac{dv}{m(v)} < z \bigg| m(1) = 0 \right).
\]

**Fact 2.3.** (Hu [16]) For \( 0 < z \leq 1 \)

\[
c_2 \exp \left(-\frac{c_3}{z^2}\right) \leq \mathbb{P}\left( \sup_{0 \leq s \leq 1} |Y(s)| < z \right) \leq c_4 \exp \left(-\frac{c_5}{z^2}\right)
\]

with some positive constants \( c_2, c_3, c_4, c_5 \).

**Fact 2.4.** (Csörgő and Révész [12]) Assume that \( T \mapsto a_T \) is a function such that \( 0 < a_T \leq T \), and both \( a_T \) and \( T/a_T \) are non-decreasing. Then

\[
\limsup_{T \to \infty} \frac{\sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |W(t+s) - W(t)|}{\sqrt{a_T (\log(T/a_T) + \log \log T)}} = \sqrt{2}, \quad \text{a.s.}
\]

**Fact 2.5.** (Strassen [18]) If \( f \in S \) defined by (1.5), then for any partition \( x_0 = 0 < x_1 < \ldots < x_k < x_{k+1} = 1 \) we have

\[
\sum_{i=1}^{k+1} \frac{(f(x_i) - f(x_{i-1}))^2}{x_i - x_{i-1}} \leq 1.
\]

**Fact 2.6.** (Chung [6])

\[
\liminf_{t \to \infty} \sqrt{\frac{\log \log t}{t}} \sup_{0 \leq s \leq t} |W(s)| = \frac{\pi}{\sqrt{8}}, \quad \text{a.s.}
\]

Define \( g(T) := \max\{s \leq T : W(s) = 0\} \). A joint lower class result for \( g(T) \) and \( M(T) := \sup_{0 \leq s \leq t} |W(s)| \) reads as follows.
Fact 2.7. (Grill [15]) Let $\beta(t), \gamma(t)$ be positive functions slowly varying at infinity, such that $0 < \beta(t) \leq 1$, $0 < \gamma(t) \leq 1$, $\beta(t)$ is non-increasing, $\beta(t)\sqrt{t} \uparrow \infty$, $\gamma(t)$ is monotone, $\gamma(t) t \uparrow \infty$, $\gamma(t)/\beta^2(t)$ is monotone. Then

$$
P\left( M(T) \leq \beta(T)\sqrt{T}, g(T) \leq \gamma(T) T \ i.o. \right) = 0 \ or \ 1$$

according as $I(\beta, \gamma) < \infty$ or $= \infty$, where

$$I(\beta, \gamma) = \int_1^\infty \frac{1}{t\beta^2(t)} \left( 1 + \frac{\beta^2(t)}{\gamma(t)} \right)^{-1/2} \exp \left( -\frac{4 - 3\gamma(t)\pi^2}{8\beta^2(t)} \right) \ dt.$$ 

Now define $d(T) := \min\{s \geq T : W(s) = 0\}$. Since $\{d(T) > t\} = \{g(t) < T\}$, we deduce from Fact 2.7 the following estimate on $d(T)$ when $T \to \infty$.

Fact 2.8. With probability 1

$$d(T) = O(T(\log T)^3), \quad T \to \infty.$$ 

3. Probability estimates

Lemma 3.1. For $T \geq 1, \delta, z > 0$ we have

$$
P\left( \sup_{0 \leq t \leq T-1} \sup_{0 \leq s \leq 1} |Y(t+s) - Y(t)| > z \right) \leq c_6 \left( \sqrt{T} \exp \left( -\frac{z^2}{8(1+\delta)} \right) + T \exp \left( -\frac{z^2}{2(1+\delta)} \right) \right)$$

with some positive constant $c_6 = c_6(\delta)$.

For the proof see Csáki et al. [7], Lemma 2.8.

Lemma 3.2. For $T > 1, 0 < \delta < 1/2, z > 1$ we have

$$
P\left( \sup_{0 \leq t \leq T-1} (Y(t+1) - Y(t)) \geq z \right) \geq \min \left( \frac{1}{2}, \frac{c_7 \sqrt{T-1}}{z} \exp \left( -\frac{z^2}{8(1-\delta)} \right) \right) - \exp (-z^2)$$

with some positive constant $c_7 = c_7(\delta) > 0$.

Proof. Let us construct an increasing sequence of stopping times by $\eta_0 := 0$ and

$$\eta_{k+1} := \inf\{t > \eta_k + 1 : W(t) = 0\}, \quad k = 0, 1, 2, \ldots$$
Let
\[ \nu_t := \min\{i \geq 1 : \eta_i > t\} \]
\[ Z_i := Y(\eta_{i-1} + 1) - Y(\eta_{i-1}), \quad i = 1, 2, \ldots \]
Then \((Z_i, \eta_i - \eta_{i-1})_{i \geq 1}\) are i.i.d. random vectors with
\[ \eta_i - \eta_{i-1} \overset{\text{law}}{=} 1 + \tau^2, \quad Z_i \overset{\text{law}}{=} Y(1), \]
where \(\tau\) has Cauchy distribution. Clearly, for \(t > 0\),
\[ \sup_{0 \leq s \leq t}(Y(s + 1) - Y(s)) \geq \max_{1 \leq i \leq \nu_t} Z_i = Z_{\nu_t}, \]
with \(Z_k := \max_{1 \leq i \leq k} Z_i\). First consider the Laplace transform \((\lambda > 0)\):
\[
\lambda \int_0^\infty e^{-\lambda u} \mathbb{P}(Z_{\nu_u} < z) \, du
\]
\[ = \lambda \sum_{k=1}^{\infty} \mathbb{E} \int_0^\infty e^{-\lambda u} 1_{(\eta_{k-1} \leq u < \eta_k)} 1_{(Z_k < z)} \, du \]
\[ = \sum_{k=1}^{\infty} \mathbb{E} \left( \left[ e^{-\lambda \eta_{k-1}} - e^{-\lambda \eta_k} \right] 1_{(Z_k < z)} \right) \]
\[ = \sum_{k=1}^{\infty} \left( \mathbb{E} \left[ 1_{(Z_k < z)} e^{-\lambda \eta_{k-1}} \right] - \mathbb{E} \left[ 1_{(Z_k < z)} e^{-\lambda \eta_k} \right] \right) \]
\[ = \sum_{k=1}^{\infty} \left( \mathbb{E} \left[ 1_{(Z_{k-1} < z)} e^{-\lambda \eta_{k-1}} \right] - \mathbb{E} \left[ 1_{(Z_{k-1} < z, Z_k \geq z)} e^{-\lambda \eta_{k-1}} \right] - \mathbb{E} \left[ 1_{(Z_k < z)} e^{-\lambda \eta_k} \right] \right) \]
\[ = 1 - \sum_{k=1}^{\infty} \mathbb{E} \left[ 1_{(Z_{k-1} < z, Z_k \geq z)} e^{-\lambda \eta_{k-1}} \right] \]
\[ = 1 - \sum_{k=1}^{\infty} \left( \mathbb{E} \left[ 1_{(Z_{k-1} < z)} e^{-\lambda \eta_{k-1}} \right] \right)^{k-1} \mathbb{P}(Y(1) \geq z) \]
\[ = 1 - \sum_{k=1}^{\infty} \left( \mathbb{E} \left[ 1_{(Z_{k-1} < z)} e^{-\lambda \eta_{k-1}} \right] \right)^{k-1} \mathbb{P}(Y(1) \geq z) \]
\[ = 1 - \frac{\mathbb{P}(Y(1) \geq z)}{1 - \mathbb{E} \left[ 1_{(Z_{1} < z)} e^{-\lambda \eta_{1}} \right]}, \]
\[ \]
\[ \text{i.e.,} \]
\[ \lambda \int_0^\infty e^{-\lambda u} \mathbb{P}(Z_{\nu_u} \geq z) \, du = \frac{\mathbb{P}(Y(1) \geq z)}{1 - \mathbb{E} \left[ 1_{(Z_{1} < z)} e^{-\lambda \eta_{1}} \right]} \]
But (recalling that \(Z_1 = Y(1)\))
\[ 1 - \mathbb{E} \left[ 1_{(Z_{1} < z)} e^{-\lambda \eta_{1}} \right] \leq 1 - \mathbb{E}(e^{-\lambda \eta_{1}}) + \mathbb{P}(Y(1) \geq z) \]
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and (cf. [14], 3.466/1)

\[ 1 - Ee^{-\lambda z} = 1 - \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\lambda (1+x^2)} \frac{dx}{1+x^2} = \frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{\lambda}} e^{-x^2} \, dx \leq 2\sqrt{\lambda}, \]

hence

\[ \lambda \int_{0}^{\infty} e^{-\lambda u} \mathbb{P} \left( Z_{\nu_u} \geq z \right) \, du \geq \frac{\mathbb{P}(Y(1) \geq z)}{2\sqrt{\lambda} + \mathbb{P}(Y(1) \geq z)}. \]

On the other hand, for any \( u_0 > 0 \) we have

\[
\lambda \int_{0}^{\infty} e^{-\lambda u} \mathbb{P} \left( Z_{\nu_u} \geq z \right) \, du = \lambda \int_{0}^{u_0} e^{-\lambda u} \mathbb{P} \left( Z_{\nu_u} \geq z \right) \, du + \lambda \int_{u_0}^{\infty} e^{-\lambda u} \mathbb{P} \left( Z_{\nu_u} \geq z \right) \, du \\
\leq \mathbb{P} \left( Z_{\nu_{u_0}} \geq z \right) + e^{-\lambda u_0}.
\]

It turns out that

\[
(3.4) \quad \mathbb{P} \left( Z_{\nu_{u_0}} \geq z \right) \geq \frac{\mathbb{P}(Y(1) \geq z)}{2\sqrt{\lambda} + \mathbb{P}(Y(1) \geq z)} - e^{-\lambda u_0} \geq \min \left( \frac{1}{2}, \frac{\mathbb{P}(Y(1) \geq z)}{4\sqrt{\lambda}} \right) - e^{-\lambda u_0},
\]

where the inequality

\[
\frac{x}{y+x} \geq \min \left( \frac{1}{2}, \frac{x}{2y} \right), \quad x > 0, \ y > 0
\]

was used. Choosing \( u_0 = T - 1 \), \( \lambda = z^2/u_0 \), and applying (2.5) of Fact 2.1, we finally get

\[
(3.5) \quad \mathbb{P} \left( \sup_{0 \leq t \leq T-1} (Y(t+1) - Y(t)) \geq z \right) \\
\geq \min \left( \frac{1}{2}, \frac{c_8(\delta) \sqrt{T-1}}{z} \exp \left( -\frac{z^2}{8(1-\delta)} \right) \right) - \exp \left( -z^2 \right).
\]

This proves Lemma 3.2. \( \square \)

**Lemma 3.3.** For \( T \geq 2 \), \( 0 \leq \kappa < 1 \) and \( \delta, z > 0 \) we have

\[
(3.6) \quad \mathbb{P} \left( \sup_{0 \leq t \leq T-1} (Y(t+1) - Y(t)) < z \right) \leq \frac{5}{T^{\kappa/2}} + \exp \left( -c_9T^{(1-\kappa)/2}e^{-(1+\delta)z^2/8} \right)
\]

with some positive constant \( c_9 = c_9(\delta) \).

See Csáki et al. [7], Lemma 3.1.

**Lemma 3.4.** For \( T \geq 1 \), \( 0 < z \leq 1/2 \) we have

\[
(3.7) \quad \mathbb{P} \left( \sup_{0 \leq t \leq T-1} \sup_{0 \leq s \leq 1} |Y(t+s) - Y(t)| < z \right) \geq \frac{c_{10}}{\sqrt{T}} \exp \left( -\frac{c_{11}}{z^2} \right)
\]

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with some positive constants $c_{10}, c_{11}$.

**Proof.** Define the events

$$ A := \left\{ \sup_{0 \leq s \leq 1} |Y(s)| < \frac{z}{4}, W(1) \geq \frac{4}{z}, \inf_{1 \leq u \leq T} W(u) \geq \frac{2}{z} \right\} $$

and

$$ \tilde{A} := \left\{ \sup_{0 \leq t \leq T-1} \sup_{0 \leq s \leq 1} |Y(t + s) - Y(t)| < z \right\}. $$

Then $A \subset \tilde{A}$, since if $A$ occurs and $t < 1$, $t + s \leq 1$, then

$$ |Y(t + s) - Y(t)| \leq 2 \sup_{0 \leq s \leq 1} |Y(s)| \leq \frac{z}{2} < z. $$

If $A$ occurs and $t < 1$, $s \leq 1$, $1 < t + s \leq T$, then

$$ |Y(t + s) - Y(t)| \leq Y(t + s) - Y(1) + |Y(t) - Y(1)| \leq \int_t^{t+s} \frac{du}{W(u)} + \frac{z}{2} < z. $$

Moreover, if $A$ occurs and $1 \leq t$, $s \leq 1$, $t + s \leq T$, then

$$ |Y(t + s) - Y(t)| = \int_t^{t+s} \frac{du}{W(u)} \leq \frac{z}{2} < z. $$

Hence $A \subset \tilde{A}$ as claimed. But by the Markov property of $W$,

$$ (3.8) \quad \mathbb{P}(A) = \int_{1/z}^{\infty} \mathbb{P}\left( \sup_{0 \leq s \leq 1} |Y(s)| < \frac{z}{4}, W(1) = x \right) \mathbb{P}\left( \inf_{1 \leq u \leq T} W(u) \geq \frac{2}{z}, W(1) = x \right) \varphi(x) \, dx, $$

where $\varphi$ denotes the standard normal density function.

Using reflection principle and $x \geq 4/z$, $z \leq 1/2$, we get

$$ (3.9) \quad \mathbb{P}\left( \inf_{1 \leq u \leq T} W(u) \geq \frac{2}{z}, W(1) = x \right) = 2 \Phi\left( \frac{x - 2/z}{\sqrt{T-1}} \right) - 1 \geq 2 \Phi\left( \frac{4}{\sqrt{T}} \right) - 1 \geq \frac{c_{12}}{\sqrt{T}}, $$

with some constant $c_{12} > 0$, where $\Phi(\cdot)$ is the standard normal distribution function. Hence

$$ (3.10) \quad \mathbb{P}(\tilde{A}) \geq \mathbb{P}(A) \geq \frac{c_{12}}{\sqrt{T}} \mathbb{P}\left( \sup_{0 \leq s \leq 1} |Y(s)| \leq \frac{z}{4}, W(1) \geq \frac{4}{z} \right). $$

To get a lower bound of the probability on the right-hand side, define $g$, $(m(v), 0 \leq v \leq 1)$, $(B(u), 0 \leq u \leq 1)$ by (2.1), (2.2) and (2.3), respectively. Recall (see Fact 2.1) that these three objects are independent, $g$ has arc sine distribution, $m$ is a Brownian meander and $B$ is a Brownian
bridge. Moreover, \((g, m, B)\) are independent of \(\text{sgn}(W(1))\) which is a Bernoulli variable. Observe that

\[
\sup_{0 \leq s \leq g} |Y(s)| = \sqrt{g} \sup_{0 \leq s \leq 1} \left| \int_0^s \frac{du}{B(u)} \right|, \\
\sup_{g \leq s \leq 1} |Y(s)| = |Y(1) - Y(g)| = \sqrt{1 - g} \int_0^1 \frac{dv}{m(v)}, \\
|W(1)| = \sqrt{1 - g} m(1).
\]

Then

\[
P\left( \sup_{0 \leq s \leq 1} |Y(s)| \leq \frac{z}{4}, W(1) \geq \frac{4}{z} \right) \\
\geq P\left( \sup_{0 \leq s \leq g} |Y(s)| \leq \frac{z}{8}, Y(1) - Y(g) \leq \frac{z}{8}, W(1) \geq \frac{4}{z} \right) \\
\geq P\left( \sqrt{g} \sup_{0 \leq s \leq 1} \left| \int_0^s \frac{du}{B(u)} \right| \leq \frac{z}{8}, \sqrt{1 - g} \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8}, \sqrt{1 - g} m(1) \geq \frac{4}{z}, W(1) > 0, g < z^2 \right) \\
\geq P\left( \sup_{0 \leq s \leq 1} \int_0^s \frac{du}{B(u)} \leq \frac{1}{8}, \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8}, m(1) \geq \frac{4}{z \sqrt{1 - z^2}}, W(1) > 0, g < z^2 \right) \\
= \frac{1}{8} \sup_{0 \leq s \leq 1} \int_0^s \frac{du}{B(u)} \leq \frac{1}{8} \sup_{0 \leq s \leq 1} \int_0^s \frac{du}{B(u)} \leq \frac{1}{8} \left( \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8}, m(1) \geq \frac{4}{z \sqrt{1 - z^2}} \right) \frac{W(1) > 0}{P(g < z^2)} \\
\geq c_{13} z P\left( \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8}, m(1) \geq \frac{4}{z \sqrt{1 - z^2}} \right) \\
= c_{13} z \int_{4/(2 \sqrt{1 - z^2})}^{\infty} P\left( \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8}, m(1) = x \right) \frac{P(1) \in dx).}
\]

It follows from Facts 2.1 and 2.2 that for \(x > 0, z > 0\)

\[ (3.11) \quad P\left( \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8}, m(1) = x \right) \geq \frac{1}{8} \left( \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8}, m(1) = 0 \right) \geq c_{14} \frac{z}{3} \exp \left( -c_{15} \frac{z}{2^2} \right) \]

and

\[ (3.12) \quad P\left( m(1) > \frac{4}{z \sqrt{1 - z^2}} \right) = \exp \left( -\frac{8}{z^2 (1 - z^2)} \right). \]

Putting (3.10), (3.11), (3.12) together, we get (3.7). \(\Box\)

**Lemma 3.5.** For \(T > 1, 0 < z \leq 1/2, 0 < \delta \leq 1/2\) we have

\[ \left(3.13\right) \quad P\left( \inf_{0 \leq t \leq T} \sup_{-1 \leq s \leq 1} |Y(t + s) - Y(t)| < z \right) \leq c_{16} \left( \exp \left( -\frac{(1 - \delta)^2}{2 (1 + \delta)^2 z^2 T} \right) + \exp \left( -\frac{c_5 \delta}{4 (1 + \delta)^2 z^2} \right) + \exp \left( c_{17} \frac{z^2}{z^2} - c_{18} \frac{z^2}{z} e^{c_{19} / z^2} \right) \right) \]

\[-935-\]
with some positive constants \( c_{16}, c_{17} = c_{17}(\delta), c_{18} = c_{18}(\delta), c_{19} = c_{19}(\delta) \).

**Proof.** Consider a positive integer \( N \) to be given later, \( h = (T - 1)/N \), \( t_k = kh, k = 0, 1, 2, \ldots, N \).

Then for \( 0 < \delta \leq 1/2 \) we have

\[
\mathbb{P} \left( \inf_{0 \leq t \leq T-1} \sup_{0 \leq s \leq 1} |Y(t+s) - Y(t)| < z \right)
\leq \mathbb{P} \left( \inf_{0 \leq k \leq N} \sup_{0 \leq s \leq 1} |Y(t_k+s) - Y(t_k)| \leq (1 + \delta)z \right) + \mathbb{P} \left( \sup_{0 \leq t \leq T-1} \sup_{0 \leq s \leq h} |Y(t+s) - Y(t)| > \delta z \right)
=: P_1 + P_2.
\]

By scaling and Lemma 3.1

\[
P_2 = \mathbb{P} \left( \sup_{0 \leq t \leq (T-1)/h} \sup_{0 \leq s \leq 1} |Y(t+s) - Y(t)| > \frac{\delta z}{\sqrt{h}} \right)
\leq c_6 \left( \sqrt{\frac{T-1}{h}} + 1 \exp \left( -\frac{\delta^2 z^2}{8h(1+\delta)} \right) \right.
\left. + (\frac{T-1}{h} + 1) \exp \left( -\frac{\delta^2 z^2}{2h(1+\delta)} \right) \right)
\leq 2c_6(N+1) \exp \left( -\frac{\delta^2 z^2}{8h(1+\delta)} \right).
\]

To bound \( P_1 \), we denote by \( d(t) := \inf\{s \geq t : W(s) = 0\} \) the first zero of \( W \) after \( t \). Consider those \( k \) for which \( \sup_{0 \leq s \leq 1} |Y(t_k+s) - Y(t_k)| \leq (1 + \delta)z \). If, moreover, \( d(t_k) \geq t_k + 1 - \delta \), which means that the Brownian motion \( W \) does not change sign over \([t_k, t_k + 1 - \delta]\), then

\[
(1 + \delta)z \geq |Y(t_k + 1 - \delta) - Y(t_k)| = \int_0^{1-\delta} \frac{ds}{|W(t_k + s)|} \geq \frac{1 - \delta}{\sup_{0 \leq s \leq T} |W(s)|},
\]

and it follows that

\[
P_1 \leq \mathbb{P} \left( \sup_{0 \leq s \leq T} |W(s)| > \frac{(1 - \delta)}{z(1+\delta)} \right)
+ \mathbb{P} \left( \exists k \leq N : \sup_{0 \leq s \leq 1} |Y(t_k+s) - Y(t_k)| \leq (1 + \delta)z; d(t_k) < t_k + 1 - \delta \right)
\leq 4 \exp \left( -\frac{(1 - \delta)^2}{2(1+\delta)^2z^2T} \right)
+ \sum_{k=0}^{N} \mathbb{P} \left( \sup_{0 \leq s \leq 1} |Y(t_k+s) - Y(t_k)| \leq (1 + \delta)z; d(t_k) < t_k + 1 - \delta \right).
\]

Let \( \hat{W}(s) = W(d(t_k) + s) \) for \( s \geq 0 \) and \( \hat{Y}(s) \) be the associated principal values. Observe that on \( \{ \sup_{0 \leq s \leq 1} |Y(t_k+s) - Y(t_k)| \leq (1 + \delta)z; d(t_k) < t_k + 1 - \delta \} \), we have \( \sup_{0 \leq u \leq \delta} |\hat{Y}(u) + (Y(d(t_k)) - Y(t_k))| < (1 + \delta)z \), and \( |Y(d(t_k)) - Y(t_k)| \leq (1 + \delta)z \) which imply that

\[
\sup_{0 \leq u \leq \delta} |\hat{Y}(u)| < 2(1 + \delta)z.
\]
By scaling and Fact 2.3 we have
\[
P \left( \sup_{0 \leq u \leq \delta} |\tilde{Y}(u)| < 2(1 + \delta) \right) \leq c_4 \exp \left( -\frac{c_5 \delta}{4(1 + \delta)^2 z^2} \right).
\]
Therefore, we obtain:
\[
P_1 \leq 4 \exp \left( -\frac{(1 - \delta)^2}{2(1 + \delta)^2 z^2 T} \right) + c_4 (N + 1) \exp \left( -\frac{c_5 \delta}{4(1 + \delta)^2 z^2} \right).
\]
Hence
\[
P_1 + P_2 \leq 4 \exp \left( -\frac{(1 - \delta)^2}{2(1 + \delta)^2 z^2 T} \right) + c_4 (N + 1) \exp \left( -\frac{c_5 \delta}{4(1 + \delta)^2 z^2} \right)
+ 2c_6 (N + 1) \exp \left( -\frac{\delta^2 z^2}{8h(1 + \delta)} \right).
\]
By taking \( N = \left[ e^{c_5 \delta/(4(1+\delta)^2 z^2)} \right] + 1 \), we get
\[
P_1 + P_2 \leq c_{16} \left( \exp \left( -\frac{(1 - \delta)^2}{2(1 + \delta)^2 z^2 T} \right) \right.
+ \exp \left( -\frac{c_5 \delta}{4(1 + \delta)^2 z^2} \right) + \exp \left( \frac{c_{17} z^2 - c_{18} z^2 T}{c_{19}/z^2} \right) \right)
\]
with relevant constants \( c_{16} \), \( c_{17} \), \( c_{18} \), \( c_{19} \), proving (3.13).

4. Proof of Theorem 1.1(i)

The upper estimation, i.e.
\[
(4.1) \quad \limsup_{T \to \infty} \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| \leq 1, \quad \text{a.s.}
\]
follows easily from Wen’s Theorem E.

Now we prove the lower bound, i.e.
\[
(4.2) \quad \limsup_{T \to \infty} \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| \geq 1, \quad \text{a.s.}
\]
In the case when \( a_T = T \), (4.2) follows from the law of the iterated logarithm (1.3) of Theorem A. Now we assume that \( a_T/T \leq \rho < 1 \), with some constant \( \rho \) for all \( T > 0 \).

By scaling, (3.2) of Lemma 3.2 is equivalent to
\[
P \left( \sup_{0 \leq t \leq T - a} (Y(t + a) - Y(t)) \geq z \sqrt{a} \right)
\[
(4.3) \geq \min \left( \frac{1}{2}, \frac{c_7 \sqrt{T/a - 1}}{z} \exp \left( -\frac{z^2}{8(1 - \delta)} \right) \right) - \exp (-z^2)
\]
for $0 < a < T$, $0 < \delta < 1/2$, $z > 1$.

Define the sequences

\[(4.4) \quad t_k := e^{7k \log k}, \quad k = 1, 2, \ldots\]

and $\theta_0 := 0$,

\[(4.5) \quad \theta_k := \inf\{t > T_k : W(t) = 0\}, \quad k = 1, 2, \ldots,\]

where $T_k := \theta_{k-1} + t_k$. For $0 < \delta < \min(1/2, 1 - \rho)$ define the events

\[A_k := \left\{ \sup_{0 \leq t \leq t_k(1-\delta) - at_k} (Y(\theta_{k-1} + t + at_k) - Y(\theta_{k-1} + t)) \geq (1 - \delta) \beta_k \right\}, \quad k = 1, 2, \ldots\]

with

\[\beta_k := \sqrt{8at_k \left( \log \sqrt{\frac{t_k}{at_k}} + \log \log t_k \right)}.
\]

Applying (4.3) with $T = t_k(1 - \delta)$, $a = at_k$, $z = (1 - \delta)\sqrt{8(\log \sqrt{t_k/at_k} + \log \log t_k)}$, we have for $k$ large

\[\mathbb{P}(A_k) = \mathbb{P} \left( \sup_{0 \leq t \leq t_k(1-\delta) - at_k} (Y(t + at_k) - Y(t)) \geq (1 - \delta) \beta_k \right) \geq \min \left( \frac{1}{2}, \frac{b_k}{(\log t_k)^{1-\delta}} \right) - \frac{1}{(\log t_k)^{8(1-\delta)^2}}\]

with

\[b_k = \frac{c_7 \sqrt{t_k(1 - \delta)/at_k - 1}}{(t_k/at_k)^{(1-\delta)/2} \sqrt{\log \sqrt{t_k/at_k} + \log \log t_k}} \geq \frac{c_20}{\sqrt{\log k}}.
\]

Hence $\sum_k \mathbb{P}(A_k) = \infty$ and since $A_k$ are independent, Borel-Cantelli lemma yields

\[\mathbb{P}(A_k \text{ i.o.}) = 1.
\]

It follows that

\[(4.6) \quad \limsup_{k \to \infty} \sup_{0 \leq t \leq t_k(1-\delta) - at_k} \frac{Y(\theta_{k-1} + t + at_k) - Y(\theta_{k-1} + t)}{\sqrt{8at_k \left( \log \sqrt{\frac{t_k}{at_k}} + \log \log t_k \right)}} \geq 1 - \delta, \quad \text{a.s.}
\]

It can be seen (cf. [9]) that we have almost surely for large enough $k$

\[t_k \leq T_k \leq t_k \left( 1 + \frac{1}{k} \right),
\]
consequently
\[ \lim_{k \to \infty} \frac{t_k}{T_k} = 1, \quad \text{a.s.} \]

Since by our assumptions
\[ t_k \leq \frac{a_t}{aT_k} \leq 1, \]
we have also
\[ \lim_{k \to \infty} \frac{a_t}{aT_k} = 1, \quad \text{a.s.} \]

On the other hand, for any \( \delta > 0 \) small enough we have almost surely for large \( k \)
\[ aT_k \leq (1 + \delta) aT_k \leq t_k \delta + aT_k, \]
thus
\[ T_k - aT_k \geq T_k - t_k \delta - aT_k, \]
consequently
\[ \sup_{0 \leq t \leq T - aT_k} \sup_{0 \leq s \leq aT_k} \left| Y(t + s) - Y(t) \right| \geq \sup_{0 \leq t \leq t_k(1-\delta) - aT_k} \left( Y(\theta_{k-1} + t + aT_k) - Y(\theta_{k-1} + t) \right), \]
hence we have also
\[ \limsup_{k \to \infty} \frac{\sup_{0 \leq t \leq T - aT_k} \sup_{0 \leq s \leq aT_k} \left| Y(t + s) - Y(t) \right|}{\sqrt{8aT_k \left( \log \frac{T_k}{aT_k} + \log \log t_k \right)}} \geq 1 - \delta, \quad \text{a.s.} \]
and since \( \delta > 0 \) can be arbitrary small, (4.2) follows by combining (4.7), (4.8), (4.9) and (4.10). \( \Box \)

5. Proof of Theorem 1.1(ii)

First assume that
\[ a_T > \frac{T}{(\log T)^\alpha} \quad \text{for some} \quad \alpha < 2. \]

By Theorem C,
\[ \liminf_{T \to \infty} \sqrt{\frac{\log \log T}{aT}} \sup_{0 \leq t \leq T - aT} \sup_{0 \leq s \leq aT} \left| Y(t + s) - Y(t) \right| \geq \liminf_{T \to \infty} \sqrt{\frac{\log \log aT}{aT}} \sup_{0 \leq s \leq aT} \left| Y(s) \right| \geq K_1, \quad \text{a.s.} \]
proving the lower bound in (1.12).

To get an upper bound, note that by scaling, (3.7) of Lemma 3.4 is equivalent to

\[(5.3) \quad P \left( \sup_{0 \leq t \leq T - a} \sup_{0 \leq s \leq a} |Y(s + t) - Y(t)| < z \sqrt{a} \right) \geq c_{10} \sqrt{\frac{a}{T}} \exp \left( - \frac{c_{11}}{z^2} \right) \]

for \( T \geq a, \ 0 < z \leq 1/2. \)

Let \( t_k \) and \( \theta_k \) be defined by (4.4) and (4.5), resp., \( T_k = \theta_k - 1 + t_k \) as in the proof of Theorem 1.1(i). Let \( c_{11} \) be the constant as in (5.3) and choose \( \delta > 0 \) such that \( \alpha/2 + c_{11}/\delta^2 < 1. \) For \( \varepsilon > 0 \) define the events

\[ E_k := \left\{ \sup_{0 \leq t \leq (1+\varepsilon)t_k - a_{t_k}(1+\varepsilon)} \sup_{0 \leq s \leq a_{t_k}(1+\varepsilon)} |Y(\theta_k - 1 + t + s) - Y(\theta_k - 1 + t)\leq \delta \right\} \sqrt{\frac{a_{t_k}}{\log \log t_k}} \].

Then putting \( T = (1 + \varepsilon)t_k, \ a = a_{(1+\varepsilon)t_k}, \ z = \delta/\sqrt{\log \log t_k} \) into (5.3), we get

\[ P(E_k) = P \left( \sup_{0 \leq t \leq (1+\varepsilon)t_k - a_{t_k}(1+\varepsilon)} \sup_{0 \leq s \leq a_{t_k}(1+\varepsilon)} |Y(t + s) - Y(t)| \leq \delta \right) \frac{\sqrt{a_{t_k}}}{\log \log t_k} \]

\[ \geq c_{10} \sqrt{\frac{a_{t_k}}{t_k}} \exp \left( -c_{11}/\delta^2 \right) \log \log t_k \geq \frac{c_{10}}{(7k \log k)^{\alpha/2 + c_{11}/\delta^2}}, \]

hence \( \sum_k P(E_k) = \infty, \) and since \( E_k \) are independent, we have \( P(E_k \text{ i.o.}) = 1, \) i.e.

\[(5.4) \lim \inf_{k \to \infty} \sqrt{\frac{\log \log t_k}{a_{t_k}}} \sup_{0 \leq t \leq (1+\varepsilon)t_k - a_{t_k}(1+\varepsilon)} \sup_{0 \leq s \leq a_{t_k}(1+\varepsilon)} |Y(\theta_k - 1 + t + s) - Y(\theta_k - 1 + t)| \leq \delta, \ a.s. \]

for any \( \varepsilon > 0. \) For large enough \( k \) by (4.7) and (4.8) we have \( a_{T_k} \leq (1+\varepsilon)a_{t_k}, \) a.s. and \( T_k - a_{T_k} \leq \theta_k - 1 + (1+\varepsilon)t_k - (1+\varepsilon)a_{t_k}, \) a.s. Thus given any \( \varepsilon > 0, \) we have for large \( k \)

\[(5.5) \sup_{0 \leq t \leq T_k - a_{T_k}} \sup_{0 \leq s \leq a_{T_k}} |Y(t + s) - Y(t)| \leq 2 \sup_{0 \leq t \leq \theta_k - 1} |Y(t)| + \sup_{0 \leq t \leq (1+\varepsilon)t_k - a_{t_k}(1+\varepsilon)} \sup_{0 \leq s \leq a_{t_k}(1+\varepsilon)} |Y(\theta_k - 1 + t + s) - Y(\theta_k - 1 + t)|. \]

By Theorem A, Fact 2.8, (4.7), (5.1) and simple calculation,

\[(5.6) \sup_{0 \leq t \leq \theta_k - 1} |Y(t)| = O(\theta_k - 1 \log \log \theta_k - 1)^{1/2} \]

\[= O(t_k - 1)^{3} \log \log t_k)^{1/2} = o \left( \frac{a_{t_k}}{\log \log t_k} \right)^{1/2}, \ a.s. \]

as \( k \to \infty. \) Assembling (5.4), (5.5) and (5.6), we get

\[ \lim \inf_{k \to \infty} \sqrt{\frac{\log \log t_k}{a_{t_k}}} \sup_{0 \leq t \leq T_k - a_{T_k}} \sup_{0 \leq s \leq a_{T_k}} |Y(t + s) - Y(t)| \]
\[
= \liminf_{k \to \infty} \sqrt{\frac{\log \log T_k}{aT_k}} \sup_{0 \leq t \leq T - aT_k} \sup_{0 \leq s \leq aT_k} |Y(t + s) - Y(t)| \leq \delta, \quad \text{a.s.}
\]

which together with (5.2) yields (1.12).

Now assume that

\[(5.7) \quad a_T \leq \frac{T}{(\log T)^\alpha} \quad \text{for some } \alpha > 2.\]

By Theorem 1.1(i),

\[
\liminf_{T \to \infty} \frac{\sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)|}{\sqrt{a_T \log(T/a_T)}} \\
\leq \limsup_{T \to \infty} \frac{\sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)|}{\sqrt{a_T \log(T/a_T)}} \\
\leq \limsup_{T \to \infty} \frac{\sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)|}{\sqrt{\frac{2a_T}{\alpha + 2} \left( \log \frac{T}{a_T} + \log \log T \right)}} \leq 2 \sqrt{\frac{\alpha + 2}{\alpha}},
\]

i.e., an upper bound in (1.13) follows.

To get a lower bound under (5.7), observe that by scaling, (3.6) of Lemma 3.3 is equivalent to

\[
P\left( \sup_{0 \leq t \leq T - a} (Y(t + a) - Y(t)) < z\sqrt{a} \right) \leq 5 \left( \frac{a}{T} \right)^{\kappa/2} + \exp \left( -c_9 \left( \frac{T}{a} \right)^{(1-\kappa)/2} e^{-\left(1+\delta\right)z^2/8} \right)
\]

for \(a \leq T, 0 \leq \kappa < 1, 0 < \delta, 0 < z\). Using (5.7) we get further

\[
P\left( \sup_{0 \leq t \leq T - a} (Y(t + a) - Y(t)) < z\sqrt{a} \right) \leq 5 \left( \frac{a}{T} \right)^{\kappa/2} + \exp \left( -c_9 \left( \frac{T}{a} \right)^{\alpha(1-\kappa)/2} e^{-\left(1+\delta\right)z^2/8} \right).
\]

In the case when (1.7) holds, (1.13) was proved in [7]. In other cases the proof is similar. Let

\(T_k = e^k\) and define the events

\[
F_k = \left\{ \sup_{0 \leq t \leq T_k - aT_k} (Y(t + aT_k) - Y(t)) \leq C_1 \sqrt{aT_k \log \frac{T_k}{aT_k}} \right\}
\]

with

\[
C_1 = 2 \sqrt{\frac{\alpha - 2 - 2\varepsilon\alpha}{(1+\delta)\alpha}}.
\]

By (5.9) with \(\kappa = 2/\alpha + \varepsilon\),

\[
P(F_k) \leq 5 \frac{k^{\alpha\kappa/2}}{k^{\alpha\kappa/2}} + \exp \left( -c_9 k^{\alpha((1-\kappa)/2 - (1+\delta)C_1^2/8)} \right) \leq 5 \frac{k^{1+\alpha\varepsilon/2}}{k^{1+\alpha\varepsilon/2}} + \exp \left( -c_9 k^{\alpha\varepsilon/2} \right).
\]
One can easily see that with these choices $\sum_k \mathbb{P}(F_k) < \infty$, consequently
\[
\liminf_{k \to \infty} \sup_{0 \leq t \leq T_k - aT_k} \frac{Y(t + aT_k) - Y(t)}{\sqrt{aT_k \log \frac{T_k}{aT_k}}} \geq C_1, \quad \text{a.s.,}
\]
implying also
\[
\liminf_{k \to \infty} \sup_{0 \leq t \leq T_k - aT_k} \sup_{0 \leq s \leq aT_k} |Y(t + s) - Y(t)| \geq 2\sqrt{\frac{\alpha - 2}{\alpha}}, \quad \text{a.s.,}
\]
for $\varepsilon$ can be chosen arbitrary small.

Since $\sup_{0 \leq t \leq T - aT} \sup_{0 \leq s \leq aT} |Y(t + s) - Y(t)|$ is increasing in $T$, we obtain a lower bound in (1.13). This together with the 0-1 law for Brownian motion complete the proof of Theorem 1.1(ii).

\[\square\]

6. Proof of Theorem 1.2(i)

If $a_T = T$, then (1.14) is equivalent to Theorem C. Now assume that $\rho := \lim_{T \to \infty} a_T/T < 1$.

First we prove the lower bound, i.e.
\[
\liminf_{T \to \infty} \mathbb{P}\left( \inf_{0 \leq t \leq T - aT} \sup_{0 \leq s \leq aT} |Y(t + s) - Y(t)| \geq c, \quad \text{a.s.} \right)
\]
\[
\leq c_{16} \left( \exp\left( -\frac{a(1 - \delta)^2}{2(1 + \delta)^2z^2T} \right) + \exp\left( -\frac{c_5\delta}{4(1 + \delta)^2z^2} \right) + \exp\left( \frac{c_{17}z}{z^2} - \frac{c_{18}aT^2}{T}e^{c_{19}z^2} \right) \right)
\]
for $a < T$, $0 < z \leq 1/2$, $0 < \delta \leq 1/2$.

Define the events
\[
G_k = \left\{ \inf_{0 \leq t \leq T_k - aT_k} \sup_{0 \leq s \leq aT_k} |Y(t + s) - Y(t)| < z_k \sqrt{aT_k} \right\} \quad k = 1, 2, \ldots
\]

Let $T_k = e^k$ and put $T = T_k + 1$, $a = aT_k$,
\[
z = z_k = C_2 \sqrt{\frac{aT_k}{T_k + 1 \log \log T_k + 1}}
\]
into (6.2). The constant $C_2$ will be choosen later. Denoting the terms on the right-hand side of (6.2) by $I_1$, $I_2$, $I_3$, resp., we have
\[
\mathbb{P}(G_k) \leq c_{16}(I_1^{(k)} + I_2^{(k)} + I_3^{(k)}),
\]
where

\[ I_1^{(k)} = \exp \left( -\frac{c_{21}}{C_2^2} \log \log T_{k+1} \right), \]

\[ I_2^{(k)} = \exp \left( -\frac{c_{22} T_k}{C_2^2 a_{T_k}} \log \log T_{k+1} \right), \]

\[ I_3^{(k)} = \exp \left( \frac{c_{23} T_k \log \log T_{k+1}}{C_2^2 a_{T_k}} - \frac{c_{24} C_2^2 a_{T_k}^2}{T_k^2 \log \log T_{k+1}} \right) + \frac{c_{25} T_k}{a_{T_k}} \]

with some constants \( c_{21} = c_{21}(\delta), c_{22} = c_{22}(\delta), c_{23}, c_{24}, c_{25}. \)

One can see easily that for any choice of positive \( C_2 \) and for all possible \( a_T \) (satisfying our conditions) we have \( \sum_k I_3^{(k)} < \infty \). So we show that for appropriate choice of \( C_2 \) we have also \( \sum_k I_j^{(k)} < \infty, j = 1, 2. \)

First consider the case \( 0 < \rho \). Choosing a positive \( \delta \), one can select \( C_2 < \min \left( \sqrt{c_{21}}, \sqrt{c_{22}} / \rho \right) \) and it is easy to verify that \( \sum_k I_j^{(k)} < \infty, j = 1, 2, \) hence also \( \sum_k \mathbb{P}(G_k) < \infty. \)

In the case \( \rho = 0 \) choose \( C_2 < (1 - \delta) / ((1 + \delta) \sqrt{2}). \) With this choice we have \( \sum_k I_1^{(k)} < \infty \) for arbitrary \( \delta > 0 \). Since \( \lim_{k \to \infty} (T_k / a_{T_k}) = \infty \), we have also \( \sum_k I_2^{(k)} < \infty \) and \( \sum_k \mathbb{P}(G_k) < \infty. \) The Borel-Cantelli lemma and interpolation between \( T_k \)'s finish the proof of (6.1). We have also verified that in the case \( \rho = 0 \) one can choose \( c = 1 / \sqrt{2} \) in (6.1), since \( \delta > 0 \) can be chosen arbitrary small.

Now we turn to the proof of the upper bound, i.e.

\[ \liminf_{T \to \infty} \frac{\sqrt{T \log \log T}}{a_T} \inf_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| \leq C_3, \quad \text{a.s.} \]

with some constant \( C_3. \)

If \( \rho > 0 \), then

\[ \inf_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| \leq \sup_{0 \leq s \leq a_T} |Y(s)| \leq \sup_{0 \leq s \leq T} |Y(s)| \]

and hence (6.3) with some positive constant \( C_3 \) follows from Theorem C.

If \( \rho = 0 \), then let for any \( \varepsilon > 0 \)

\[ \lambda_T := \inf \{ t : |W(t)| = \sup_{0 \leq s \leq T(1 - \varepsilon)} |W(s)| \}. \]

According to the law of the iterated logarithm, with probability one there exists a sequence \( \{ T_i, i \geq 1 \} \) such that \( \lim_{i \to \infty} T_i = \infty \) and

\[ |W(\lambda_T)| \geq \sqrt{2T_i(1 - \varepsilon) \log \log T_i}. \]
But Fact 2.4 implies that for \( \varepsilon > 0 \)

\[
(W(\lambda T_i) - W(s)) \leq \sqrt{2(1+\varepsilon)\varepsilon T_i \log \log T_i}, \quad \lambda T_i \leq s \leq \lambda T_i + \varepsilon T_i, \quad i \geq 1.
\]

Now assume that \( W(\lambda T_i) > 0 \). The case when \( W(\lambda T_i) < 0 \) is similar. Then (6.5) and (6.6) imply

\[
W(s) \geq \left( \sqrt{1-\varepsilon} - \sqrt{\varepsilon(1+\varepsilon)} \right) \sqrt{2T_i \log \log T_i}, \quad \lambda T_i \leq s \leq \lambda T_i + \varepsilon T_i.
\]

\( \rho = 0 \) implies that \( a_T \leq \varepsilon T \) for any \( \varepsilon > 0 \) and large enough \( T \), hence we have from (6.7) for large \( i \)

\[
\sup_{0 \leq s \leq a_{T_i}} (Y(\lambda T_i + s) - Y(\lambda T_i)) = Y(\lambda T_i + a_{T_i}) - Y(\lambda T_i) = \int_{\lambda T_i}^{\lambda T_i + a_{T_i}} \frac{ds}{W(s)} \leq \frac{a_{T_i}}{\left( \sqrt{1-\varepsilon} - \sqrt{\varepsilon(1+\varepsilon)} \right) \sqrt{2T_i \log \log T_i}}.
\]

Since \( \varepsilon > 0 \) is arbitrary, (6.3) follows with \( C_3 = 1/\sqrt{2} \). This completes the proof of Theorem 1.2(i).

7. Proof of Theorem 1.2(ii)

If \( \rho = 1 \), then (1.15) is equivalent to (1.3) of Theorem A. So we may assume that \( 0 < \rho < 1 \). It suffices to show (1.15) when \( a_T = \rho T \).

First we prove the upper bound

\[
\limsup_{T \to \infty} \frac{\inf_{0 \leq t \leq T-\rho T} \sup_{0 \leq s \leq \rho T} |Y(t + s) - Y(t)|}{\sqrt{8T \log \log T}} \leq \rho, \quad \text{a.s.}
\]

Let \( k \) be the largest integer for which \( k \rho < 1 \) and put \( x_i = i \rho, \ i = 0, 1, \ldots, k, \ x_{k+1} = 1 \). It suffices to show that if \( f \in \mathcal{S} \) defined by (1.5), then

\[
\min_{1 \leq i \leq k+1} |f(x_i) - f(x_{i-1})| \leq \rho.
\]

Assume on the contrary that

\[
|f(x_i) - f(x_{i-1})| > \rho, \quad \forall i = 1, 2, \ldots, k + 1.
\]

Then

\[
\sum_{i=1}^{k+1} \frac{(f(x_i) - f(x_{i-1}))^2}{x_i - x_{i-1}} > \sum_{i=1}^{k} \frac{\rho^2}{\rho} + \frac{\rho^2}{1 - k \rho} = k \rho + \frac{\rho^2}{1 - k \rho} \geq 1,
\]

contradicting (2.12) of Fact 2.5. This proves (7.1).
The lower bound

\[(7.2) \limsup_{T \to \infty} \inf_{0 \leq t \leq T - aT} \sup_{0 \leq s \leq aT} |Y(t + s) - Y(t)| \geq \rho, \quad \text{a.s.} \]

follows from the fact that by Theorem B the function \(f(x) = x, 0 \leq x \leq 1\) is a limit point of

\[
\frac{Y(xt)}{\sqrt{8T \log \log T}}
\]

and for this function

\[
\min_{0 \leq x \leq 1 - \rho} |f(x + \rho) - f(x)| = \rho.
\]

This completes the proof of Theorem 1.2(iia). \(\square\)

Now assume that

\[(7.3) \lim_{T \to \infty} \frac{aT(\log \log T)^2}{T} = 0. \]

Define \(\lambda_T\) as in (6.4). Then according to Chung’s LIL (cf. Fact 2.6)

\[(7.4) |W(\lambda_T)| \geq \frac{\pi}{\sqrt{8}} (1 - \varepsilon) \sqrt{\frac{T}{\log \log T}} \]

for \(\varepsilon > 0\) and all \(T\) sufficiently large. But according to Fact 2.4,

\[
\sup_{0 \leq s \leq aT} |W(\lambda_T + s) - W(\lambda_T)| \\
\leq \sqrt{(2 + \varepsilon)aT(\log(T/aT) + \log \log T) \leq \sqrt{(2 + \varepsilon)\varepsilon T \log \log T}. \]

Assuming \(W(\lambda_T) > 0\), on choosing suitable \(\varepsilon > 0\) we get for some \(c_{26} > 0\)

\[
W(\lambda_T + s) \geq W(\lambda_T) - \sqrt{(2 + \varepsilon)\varepsilon T \log \log T} \geq c_{26} \sqrt{\frac{T}{\log \log T}}.
\]

Hence

\[
\inf_{0 \leq t \leq T - aT} \sup_{0 \leq s \leq aT} |Y(t + s) - Y(t)| \leq Y(\lambda_T + aT) - Y(\lambda_T) \\
= \int_0^{aT} \frac{ds}{W(\lambda_T + s)} \leq \frac{aT}{c_{26}} \sqrt{\frac{\log \log T}{T}}
\]

for all large \(T\).

The case when \(W(\lambda_T) < 0\) is similar. This shows the upper bound in (1.16).
For the lower bound we use Fact 2.7: with probability one

\[
g_T \leq \frac{T}{(\log \log T)^2}, \quad \max_{0 \leq u \leq T} |W(u)| \leq \frac{\pi}{\sqrt{2}} \sqrt{\frac{T}{\log \log T}}, \quad \text{i.o.}
\]

According to Theorem 1.2(i) we have for any \( \varepsilon > 0 \) and all large \( T \)

\[
\inf_{0 \leq t \leq T(\log \log T)^{-2}} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| \geq \frac{(K_4 - \varepsilon)a_T}{\sqrt{T}} \frac{\log \log T}{\sqrt{1 + \varepsilon}T}.
\]

On the other hand, if \( T(\log \log T)^{-2} \leq t \leq T - a_T \), and (7.5) is satisfied, then

\[
|Y(t + a_T) - Y(t)| = \int_t^{t + a_T} \frac{ds}{W(s)} \geq \frac{a_T \sqrt{2} \log \log T}{\pi \sqrt{T}}, \quad \text{i.o.}
\]

Combining (7.6) and (7.7) for \( \varepsilon > 0 \) with probability one

\[
\inf_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| \geq \min \left( \frac{K_4 - \varepsilon}{\sqrt{1 + \varepsilon}}, \frac{\sqrt{2}}{\pi} \right) \frac{a_T \sqrt{\log \log T}}{T}, \quad \text{i.o.}
\]

This shows the lower bound in (1.16). The proof of Theorem 1.2(ii) is complete by applying the 0-1 law for Brownian motion. \( \square \)

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**References**


