REPRESENTATIONS OF URBANIK’S CLASSES AND MULTIPARAMETER ORNSTEIN-UHLENBECK PROCESSES

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Abstract
A class of integrals with respect to homogeneous Lévy bases on $\mathbb{R}^k$ is considered. In the one-dimensional case $k = 1$ this class corresponds to the selfdecomposable distributions. Necessary and sufficient conditions for existence as well as some representations of the integrals are given. Generalizing the one-dimensional case it is shown that the class of integrals corresponds to Urbanik’s class $L_{k-1}(\mathbb{R})$. Finally, multiparameter Ornstein-Uhlenbeck processes are defined and studied.

1 Introduction

The purpose of this note is twofold. First of all, for any integer $k \geq 1$ we study the integral

$$
\int_{\mathbb{R}^k_+} e^{-t} \; M(dt),
$$

where $M = \{M(A) : A \in \mathcal{B}_0(\mathbb{R}^k)\}$ is a homogeneous Lévy basis on $\mathbb{R}^k$ and $t = t_1 + \cdots + t_k$ is the sum of the coordinates. Recall that a homogeneous Lévy basis is an example of an independently scattered random measure as defined in [9]; see the next section for further details. The one-dimensional case $k = 1$, where $M$ is induced by a Lévy process, is very well studied; see [10] for a survey. For example, in case of existence when $k = 1$ the integral has a selfdecomposable distribution ([7, 14]) and it is thus the marginal distribution of a stationary Ornstein-Uhlenbeck process. Moreover, necessary and sufficient conditions for the existence of (1.1) for $k = 1$ are also well-known. In the present note we give necessary and sufficient conditions for the existence of (1.1) for arbitrary $k$ and provide several representations of the integral. The main result, Theorem 3.1, shows that for arbitrary $k \geq 1$ the law of (1.1) belongs to Urbanik’s class $L_{k-1}(\mathbb{R})$. 
and conversely that any distribution herein is representable as in (1.1). The proof of the main theorem is in fact very easy. It relies only on a transformation rule for random measures (see Lemma 2.1) and well-known representations of Urbanik’s classes.

Assuming that (1.1) exists we may define a process \( Y = \{ Y_t : t \in \mathbb{R}^k \} \) as

\[
Y_t = \int_{s \leq t} e^{-(t-s)} M(ds),
\]

where \( s \leq t \) should be understood coordinatewise. The second purpose of the note is to study some of the basic properties of this process. It is easily seen that \( Y \) is stationary and can be chosen lamp, where we recall that the latter is the multiparameter analogue of being càdlàg. In the case \( k = 1 \), \( Y \) is often referred to as an Ornstein-Uhlenbeck process and we shall thus call \( Y \) a \( k \)-parameter Ornstein-Uhlenbeck process. In the case \( k = 1 \), \( Y \) is representable as

\[
Y_t = Y_0 - \int_0^t Y_s \, ds + M((0, t]) \quad \text{for } t \geq 0.
\]

We give the analogous formula to this equation in the case \( k = 2 \).

In the Gaussian case, Hirsch and Song [5] gave an alternative definition of \( k \)-parameter Ornstein-Uhlenbeck processes; however, in Remark 4.2(2) we show that the two definitions give rise to the same processes.

The next section contains a few preliminary results. Section 3 concerns the main result, namely characterizations of (1.1). Finally, in Section 4 we study multiparameter Ornstein-Uhlenbeck processes.

## 2 Preliminaries

Let Leb denote Lebesgue measure on \( \mathbb{R}^k \). Throughout this note all random variables are defined on a probability space \((\Omega, \mathcal{F}, P)\). The law of a random vector is denoted by \( \mathcal{L}(X) \) and for a set \( N \) and two families \( \{ X_t : t \in N \} \) and \( \{ Y_t : t \in N \} \) of random vectors write \( \{ X_t : t \in N \} \equiv \{ Y_t : t \in N \} \) if all finite dimensional marginals are identical. Furthermore, we say that \( \{ X_t : t \in N \} \) is a modification of \( \{ Y_t : t \in N \} \) if \( X_t = Y_t \) a.s. for all \( t \in N \). Let \( \text{ID} = \text{ID}(\mathbb{R}) \) denote the class of infinitely divisible distributions on \( \mathbb{R} \). That is, a distribution \( \mu \) on \( \mathbb{R} \) is in ID if and only if

\[
\hat{\mu}(z) := \int_{\mathbb{R}} e^{izx} \mu(dx) = \exp \left\{ -\frac{1}{2} z^2 \sigma^2 + iyz + \int_{\mathbb{R}} g(z, x) v(dx) \right\} \quad \text{for all } z \in \mathbb{R},
\]

where \( g(z, x) = e^{izx} - 1 - izx \mathbf{1}_D(x), \ D = [-1, 1], \) and \((\sigma^2, v, \gamma)\) is the characteristic triplet of \( \mu \), that is, \( \sigma^2 \geq 0, \ v \) is a Lévy measure on \( \mathbb{R} \) and \( \gamma \in \mathbb{R} \). For \( t \geq 0 \) and \( \mu \) in ID, \( \mu^t \) denotes the distribution in ID with \( \hat{\mu}^t = \hat{\mu}^\gamma \).

For \( S \in \mathcal{B}(\mathbb{R}^k) \) let \( \mathcal{B}_b(S) \) be the set of bounded Borel sets in \( S \). Let \( \Lambda = \{ \Lambda(A) : A \in \mathcal{B}_b(S) \} \) denote a family of (real valued) random variables indexed by \( \mathcal{B}_b(S) \), the set of bounded Borel set in \( S \).

Following [9] we call \( \Lambda \) an independently scattered random measure on \( S \) if the following conditions are satisfied:

(i) \( \Lambda(A_1), \ldots, \Lambda(A_n) \) are independent whenever \( A_1, \ldots, A_n \in \mathcal{B}_b(S) \) are disjoint.

(ii) \( \Lambda(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \Lambda(A_n) \) a.s. whenever \( A_1, A_2, \ldots \in \mathcal{B}_b(S) \) are disjoint with \( \bigcup_{n=1}^\infty A_n \in \mathcal{B}_b(S) \). Here the series converges almost surely.
(iii) \( \mathcal{L}(\Lambda(A)) \in \mathcal{D} \) for all \( A \in \mathcal{B}(S) \).

If in addition there is a \( \mu \in \mathcal{D} \) such that \( \mathcal{L}(\Lambda(A)) = \mu^{\text{Leb}}(A) \) for all \( A \in \mathcal{B}(S) \) then \( \Lambda \) is called a homogeneous Lévy basis on \( S \) and \( \Lambda \) is said to be associated with \( \mu \). Note that in (ii) above there is a null set depending on the sequence \( A_1, A_2, \ldots \). Thus, if \( \Lambda \) is an independently scattered random it is generally not true that for \( \omega \) outside a set of probability zero the mapping \( A \to \Lambda(A)(\omega) \) is a (signed) measure when \( A \) is in a \( \sigma \)-field included in \( \mathcal{B}(S) \). Thus, \( \Lambda \) is not a random measure in the sense of [8]. The problem of finding large subsets \( \mathcal{A} \) of \( \mathcal{B}(S) \) such that the mapping \( \mathcal{A} \ni A \to \Lambda(A)(\omega) \) is regular (in some sense) for \( \omega \) outside a null set is studied in [1, 3].

Let \( \Lambda \) denote an independently scattered random measure on \( S \). Recall from [9] that there exists a control measure \( \lambda \) for \( \Lambda \) and a family of characteristic triplets \( (\sigma^2, v_i(dx), \gamma_i) \) measurable in \( s \), such that for \( A \in \mathcal{B}(S) \), \( \mathcal{L}(\Lambda(A)) \) has characteristic triplet \( (\sigma^2(A), v(A)(dx), \gamma(A)) \) given by

\[
\sigma^2(A) = \int_A \sigma^2 \lambda(ds), \quad v(A)(dx) = \int_A v_i(dx) \lambda(ds), \quad \gamma(A) = \int_A \gamma_i \lambda(ds).
\] (2.1)

If \( \Lambda \) is a homogeneous Lévy basis associated with \( \mu \) then \( \lambda \) equals \( \text{Leb} \) and \( (\sigma^2, v_i(dx), \gamma_i) = (\sigma^2, v(dx), \gamma) \) where \( (\sigma^2, v, \gamma) \) is the characteristic triplet of \( \mu \).

As an example let \( k = 1 \). If \( \Lambda \) is a homogeneous Lévy basis on \( \mathbb{R} \) associated with \( \mu \) then for all \( s \in \mathbb{R} \) the process \( \{\Lambda((s, s+ t]) : t \geq 0\} \) is a Lévy process in law in the sense of [12], p. 3. In particular it has a càdlàg modification which is a Lévy process and \( \mathcal{L}(\Lambda((s, s+ t])) = \mu \) for all \( s \in \mathbb{R} \). Conversely, if \( Z = \{Z_t : t \in \mathbb{R}\} \) is a Lévy process indexed by \( \mathbb{R} \) (i.e. it is càdlàg with stationary independent increments) then \( \Lambda = \{\Lambda(A) : A \in \mathcal{B}(\mathbb{R})\} \) defined as \( \Lambda(A) = \int_1_A dZ \) is a homogeneous Lévy basis. Similarly, a so-called natural additive process induces an independently scattered random measure; see [13].

Since an independently scattered random measure \( \Lambda \) on \( S \) does not in general induce a usual measure an \( \omega \)-wise definition of the associated integral is not possible. Therefore, integration with respect to \( \Lambda \) will always be understood in the sense developed in [9]. Recall the definition cf. page 460 in [9]:

A function \( f : S \to \mathbb{R} \) is called simple if there is an \( n \geq 1 \), \( \alpha_i \in \mathbb{R} \) and \( A_i \in \mathcal{B}(S) \) such that \( f = \sum_{i=1}^n \alpha_i 1_{A_i} \). In this case define \( \int_A f d\Lambda = \sum_{i=1}^n \alpha_i \Lambda(A_i \cap A) \). In general, if \( f : S \to \mathbb{R} \) is a measurable function then \( f \) is called \( \Lambda \)-integrable if there is a sequence \( (f_n)_{n \geq 1} \) of simple functions such that (i) \( f_n \to f \) \( \lambda \)-a.s., where \( \lambda \) is the control measure, and (ii) the sequence \( \int_A f_n d\Lambda \) converges in probability for every \( A \in \mathcal{B}(S) \). In this case the limit in (ii) is called the integral of \( f \) over \( \Lambda \) and is denoted \( \int_A f d\Lambda \). The integral is well-defined, i.e., it does not depend on the approximating sequence.

Let \( \mathcal{B}_\lambda \) denote the set of \( A \) in \( \mathcal{B}(S) \) for which \( 1_A \) is \( \Lambda \)-integrable. Then \( \mathcal{B}_\lambda \) contains \( \mathcal{B}_b(S) \) and we can extend \( \Lambda \) to \( \mathcal{B}_\lambda \) by setting

\[
\Lambda(A) = \int_1_A d\Lambda, \quad A \in \mathcal{B}_\lambda.
\]

Moreover, \( \{\Lambda(A) : A \in \mathcal{B}_\lambda\} \) is an independently scattered random measure; that is, (i)–(iii) above are satisfied when \( \mathcal{B}_a(S) \) is replaced by \( \mathcal{B}_\lambda \). For \( A \in \mathcal{B}_\lambda \), \( \Lambda(A) \) still has characteristic triplet given by (2.1).

Let \( T \in \mathcal{B}(\mathbb{R}^d) \) for some \( d \). Given an independently scattered random measure \( \Lambda \) on \( S \) and a function \( \phi : S \to T \) satisfying \( \phi^{-1}(B) \in \mathcal{B}_\lambda \) for all \( B \in \mathcal{B}_b(T) \), we can define an independently scattered random measure on \( T \), called the image of \( \Lambda \) under \( \phi \), to be denoted \( \Lambda_\phi = \{\Lambda_\phi(B) : B \in \mathcal{B}_b(T)\} \), as

\[
\Lambda_\phi(B) = \Lambda(\phi^{-1}(B)), \quad B \in \mathcal{B}_b(T).
\]
Similarly, if $\psi : S \to \mathbb{R}$ is measurable and locally bounded then $\bar{\Lambda} = \{ \bar{\Lambda}(A) : A \in \mathcal{B}(S) \}$, defined as

$$\bar{\Lambda}(A) = \int_A \psi \, d\Lambda, \quad A \in \mathcal{B}(S),$$

is an independently scattered random measure on $S$.

Keeping this notation we have the following result.

**Lemma 2.1.**  
1. Let $g : S \to \mathbb{R}$ be measurable. Then $g$ is $\bar{\Lambda}$-integrable if and only if $g \psi$ is $\Lambda$-integrable and in this case $\int_S g \, d\bar{\Lambda} = \int_S (g \psi) \, d\Lambda$.

2. Let $f : T \to \mathbb{R}$ be measurable. Assume that $(\sigma_s^2, \nu_s(\,dx\,), \gamma_s)$ in (2.1) only depends on $s$ through $\phi$; that is, there is a family of characteristic triplets $(\sigma_t^2, \nu_t(\,dx\,), \gamma_t)$ for $t \in T$, measurable in $t$, such that

$$\sigma^2(A) = \int_A \sigma_\phi^2 \, d\lambda(ds),$$

$$\nu(A)(dx) = \int_A \nu_\phi(dx) \, d\lambda(ds),$$

$$\gamma(A) = \int_A \gamma_\phi \, d\lambda(ds).$$

(2.2)

Then $f$ is $\Lambda_\phi$-integrable if and only if $f \circ \phi$ is $\Lambda$-integrable and in this case $\int_T f \, d\Lambda_\phi = \int_T (f \circ \phi) \, d\Lambda$. In general, that is, when (2.2) is not satisfied, only the if-part of the statement is true.

**Proof.** Using [9], Theorem 2.7(iv), and formula (2.1) we see that for every $A \in \mathcal{B}(S)$ the characteristic triplet $(\bar{\sigma}^2(A), \bar{\nu}(A)(dx), \bar{\gamma}(A))$ of $L(\bar{\Lambda}(A))$ is given by

$$\bar{\sigma}^2(A) = \int_A \bar{\sigma}^2 \, d\lambda(ds), \quad \bar{\nu}(A)(dx) = \int_A \bar{\nu}_s(dx) \, d\lambda(ds), \quad \bar{\gamma}(A) = \int_A \bar{\gamma}_s \, d\lambda(ds),$$

where

$$\bar{\sigma}^2_s = \psi_s^2 \sigma_s^2, \quad \bar{\nu}_s = \psi_s \nu_s + \int_{\mathbb{R}} \psi_s x [1_B(\psi_s x) - 1_B(x)] \nu_s(dx)$$

$$\bar{\gamma}_s(B) = \int_B 1_B(\psi_s x) \nu_s(dx), \quad B \in \mathcal{B}(\mathbb{R} \setminus \{0\}).$$

According to [9], Theorem 2.7, $g \psi$ is $\Lambda$-integrable if and only if the following three conditions are satisfied:

$$\int_S (\psi_s g_s)^2 \sigma_s^2 \, d\lambda(ds) < \infty$$

(2.3)

$$\int_S \int_{\mathbb{R}} (1 \wedge \psi_s g_s x^2) \nu_s(dx) \, d\lambda(ds) = \int_S \int_{\mathbb{R}} (1 \wedge [g_s x]^2) \nu_s(dx) \, d\lambda(ds) < \infty$$

(2.4)

$$\int_S |\psi_s g_s| \cdot |\gamma_s| + \int_{\mathbb{R}} x [1_B(\psi_s g_s x) - 1_B(x)] \nu_s(dx) |\lambda(ds) < \infty.$$  

(2.5)
Similarly, \( g \) is \( \Lambda \)-integrable if and only if (2.3) and (2.4) are satisfied and

\[
\int_S |g_s \tilde{\gamma}_s| + \int_{\mathbb{R}} (g_s x \varphi_s(x) - 1_B(x)) \lambda(dx) < \infty.
\]  

(2.6)

But noticing that the last integral equals

\[
\int_S |g_s \psi_s| \cdot |\gamma_s| + \int_{\mathbb{R}} x 1_B(\psi_s(x) - 1_B(x)) \nu_s(dx) \lambda(ds)
\]

\[
+ \int_{\mathbb{R}} x [1_B(\psi_s x) - 1_B(x \psi_s)] \nu_s(dx) \lambda(ds)
\]

\[
= \int_S |\psi_s g_s| \cdot |\gamma_s| + \int_{\mathbb{R}} x [1_B(\psi_s g_s x) - 1_B(x)] \nu_s(dx) \lambda(ds),
\]

one sees that (2.5) and (2.6) are equivalent. Thus we have (1).

(2): Assume (2.2). Then \( \mathcal{L}(\Lambda_\phi(B)), B \in \mathcal{B}(\mathbb{R}) \), has characteristic triplet given by

\[
\left( \int_B \sigma^2 \lambda_\phi(dt), \int_B \nu_1(dx) \lambda_\phi(ds), \int_B \gamma_1 \lambda_\phi(dt) \right),
\]

where \( \lambda_\phi \) is the image measure of \( \lambda \) under the the function \( \phi \). Thus, the first part of (2) follows from [9], Theorem 2.7, using the ordinary transformation rule.

In the general case, without (2.2), \( f \circ \phi \) is \( \Lambda \)-integrable if and only if there is a sequence of simple functions \( g_n \) approximating \( f \circ \phi \) such that \( \int_A g_n d\Lambda \) converges in probability for all \( A \in \mathcal{B}(S) \). Assume this is the case. By the explicit construction in the proof of [9], Theorem 2.7, we can choose \( g_n \) on the form \( g_n = h_n \circ f \circ \phi \) where \( h_n \) is simple. Thus, since, by definition,

\[
\int_B (h_n \circ f) d\Lambda_\phi = \int_{\phi^{-1}(B)} (h_n \circ f \circ \phi) d\Lambda
\]

for all \( n \) and all \( B \in \mathcal{B}(T) \), it follows by definition of integrability that \( f \) is \( \Lambda_\phi \)-integrable.

Before continuing we recall a few basic properties of the class of selfdecomposable distributions and the classes \( L_m \). See e.g. [6, 7, 11, 14, 15, 16] for fuller information, and [10] for a nice summary of the results used below. Let \( L_0 = L_0(\mathbb{R}) \) denote the class of selfdecomposable distributions on \( \mathbb{R} \). Recall, e.g. from [12], Theorem 15.3, that a probability measure \( \mu \) is in \( L_0 \) if and only if it is the limit in distribution as \( n \to \infty \) of variables \( a_n + b_n S_n \) where \( a_n, b_n \) are real numbers, \( S_n = \sum_{i=1}^n Z_i \) and \( (Z_i)_{i \geq 1} \) is a sequence of independent random variables. For \( m = 1, 2, \ldots \) define \( L_m = L_m(\mathbb{R}) \) recursively as follows: \( \mu \in L_m \) if and only if for all \( b > 1 \) there is a \( \rho_b \in L_{m-1} \) such that \( \tilde{\mu}(z) = \tilde{\mu}(b^{-1}z) \tilde{\rho}_b(z) \) for all \( z \in \mathbb{R} \). It is well-known (see e.g. [10], Propositions 5 and 11) that the sets \( L_m \) are decreasing in \( m \) and that the stable distributions are in \( L_m \) for all \( m \).

Let \( \text{ID}_{\log} \) denote the class of infinitely divisible distributions \( \mu \) with Lévy measure \( \nu \) satisfying

\[
\int_{|x| > 2} \log |x| \nu(dx) < \infty.
\]

It is well-known that \( \text{ID}_{\log} \) consists precisely of those \( \mu \in \text{ID} \) for which the integral \( \int_0^\infty e^{-t} dZ_t \) exists. Here \( \{Z_t : t \geq 0\} \) is a Lévy process with \( \mu = \mathcal{L}(Z_1) \). In case of existence \( \mathcal{L}(\int_0^\infty e^{-t} dZ_t) \) is in \( L_0 \). Using this, an alternative useful characterization of \( L_m \) can be formulated as follows: Let \( \Phi : \text{ID}_{\log} \to L_0 \) be given by \( \Phi(\mu) = \mathcal{L}(\int_0^\infty e^{-t} dZ_t) \) where \( Z \) is as above. Then \( \Phi \) is one-to-one and onto \( L_0 \). Moreover, for \( m = 1, 2, \ldots \) we have \( L_m = \Phi(L_{m-1} \cap \text{ID}_{\log}) \).
3 Existence and characterizations of the integral

Assume that \( M = \{M(A) : A \in \mathcal{B}(\mathbb{R}_+^k)\} \) is a homogeneous Lévy basis on \( \mathbb{R}_+^k \) associated with \( \mu \in \text{ID} \) which has characteristic triplet \((\sigma^2, \nu, \gamma)\).

Let \( f : \mathbb{R}_+^k \to \mathbb{R}_+ \) be given by \( f(t) = t_1t_2 + \cdots + t_k t_j \) and let \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) be given by \( g(x) = \frac{x^+}{x^-} \).

Then the image of \( M \) under \( f \), \( M_f = \{M_f(B) : B \in \mathcal{B}(\mathbb{R}_+)\} \), is an independently scattered random measure on \( \mathbb{R}_+ \) and \( \mathcal{L}(M_f((0,x])) = \mu^{x/t_1} \) for all \( x \geq 0 \). Since in particular

\[
\mathcal{L}(M_f([0,y])) = \mathcal{L}(M_f([0, g^{-1}(y)])) = \mu^y \quad \text{for } y \geq 0,
\]

it follows that \( M_{g,f} = \{M_{g,f}(B) : B \in \mathcal{B}(\mathbb{R}_+)\} \) is a homogeneous Lévy basis on \( \mathbb{R}_+ \) associated with \( \mu \). Writing \( \Phi^{(k)} \) for \( \Phi \circ \cdots \circ \Phi \) (\( k \) times), the main result can be formulated as follows.

**Theorem 3.1.** (1) The three integrals

\[
\int_{\mathbb{R}_+^k} e^{-t_1} M(dt), \quad \int_{\mathbb{R}_+} e^{-x} M_f(dx), \quad \int_{\mathbb{R}_+} e^{-(k_1/y)} M_{g,f}(dy) \tag{3.1}
\]

exist at the same time and are identical in case of existence. Assume existence and let \( \tilde{\mu} = \mathcal{L}(\int_{\mathbb{R}_+^k} e^{-t_1} M(dt)) \). Then \( \tilde{\mu} = \Phi^{(k)}(\mu) \in L_{k-1} \) and \( \tilde{\mu} \) has characteristic triplet \((\tilde{\sigma}^2, \tilde{\nu}, \tilde{\gamma})\) given by

\[
\tilde{\sigma}^2 = \frac{\sigma^2}{2^k} \tag{3.2}
\]

\[
\tilde{\nu}(B) = \frac{1}{(k-1)!} \int_{\mathbb{R}_+^k} \int_0^\infty s^{k-1} 1_B(e^{-s}y) ds \nu(dy), \quad B \in \mathcal{B}(\mathbb{R}) \tag{3.3}
\]

\[
\tilde{\gamma} = \gamma + \frac{1}{(k-1)!} \int_0^\infty s^{k-1} e^{-s} \int_{1<|y|<e^s} \nu(dy) ds. \tag{3.4}
\]

(2) A necessary and sufficient condition for the existence of the integrals in (3.1) is that

\[
\int_{|x| > 2} (\log |x|)^k \nu(dx) < \infty. \tag{3.5}
\]

(3) Let \( k \geq 2 \) and assume that the integrals in (3.1) exist. Then

\[
\int_{\mathbb{R}_+^k} e^{-t_1} M(dt) = \int_{\mathbb{R}_+} e^{-x} \Lambda(dx), \tag{3.6}
\]

where \( \Lambda = \{\Lambda(B) : B \in \mathcal{B}(\mathbb{R}_+)\} \) is given as

\[
\Lambda(B) = \int_{B \times \mathbb{R}_+^{k-1}} e^{-\sum_{i=2}^k t_i} M(dt), \quad B \in \mathcal{B}(\mathbb{R}_+). \tag{3.7}
\]

Moreover, \( \Lambda \) is a homogeneous Lévy basis on \( \mathbb{R}_+ \). The distribution associated with \( \Lambda \) is \( \Phi^{(k-1)}(\mu) \) which has characteristic triplet given by (3.2)–(3.4) with \( k \) replaced by \( k-1 \).
(4) Conversely, to every distribution $\tilde{\mu} \in L_{k-1}$ there is a distribution $\mu \in ID$ with characteristic triplet $(\sigma^2, \nu, \gamma)$ with $\nu$ satisfying (3.5) such that $\tilde{\mu}$ is given by $\tilde{\mu} = \mathcal{L}(\int_{\mathbb{R}^+} e^{-tM} dt)$ where $M$ is a homogeneous Lévy basis on $\mathbb{R}^+_k$ associated with $\mu$.

Proof. We can apply the first part of Lemma 2.1(2) to the first two integrals in (3.1) since $\Lambda$ is homogeneous. Likewise, the lemma applies to the last two integrals since $g$ is one-to-one. It hence follows immediately that the three integrals in (3.1) exist at the same time and are identical in case of existence. The remaining assertions, except (3), follow from Theorem 49 and Remark 58 of Rocha-Arteaga and Sato [10]. First of all, by Remark 58 a distribution is in $L_{k-1}$ if and only if it is representable as the law of the last integral in (3.1). That $\tilde{\mu} = \Phi^{(k)}(\mu)$ in case of existence follows from Remark 58 combined with Theorem 49. Using this, the result in (2) is equation (2.44) in Theorem 49, and the representation of the characteristic triplet in (1) is (2.47)-(2.49) in Theorem 49.

To prove (3) assume $t \to e^{-t}$ is $M$-integrable and let $\tilde{M}(A) = \int_A \psi(t) M(dt)$ for $A \in \mathcal{B}_b(\mathbb{R}^+_k)$ where $\psi(t) = e^{-\sum_{i=1}^k \xi_i}$. Since for $B \in \mathcal{B}_b(\mathbb{R}^+_k)$ there is a constant $c > 0$ such that $\psi(t) \int_{B \times \mathbb{R}^k} |t| \leq ce^{-\sum_{i=1}^k \xi_i} \int_{B \times \mathbb{R}^k} |t|$ we have by Lemma 2.1(1) that $B \times \mathbb{R}^k \in \mathcal{B}_\tilde{M}$. Thus, $\Lambda$ in (3.7) is well-defined and a homogeneous Lévy basis. Moreover, $\Lambda = (\tilde{M})_\phi$ where $\phi$ is $M$-integrable and homogeneous Lévy basis. Since, with $f : \mathbb{R}^k \to \mathbb{R}^k$ given by $f(x) = e^{-x}$, the mapping $f \circ \phi(t) = e^{-t}$ is $\tilde{M}$-integrable by Lemma 2.1(1) it follows from the last part of Lemma 2.1(2) that $f$ is $\Lambda$-integrable and we have (3.6). That is, $\tilde{\mu}$ in (1) is of the form $\tilde{\mu} = \mathcal{L}(\int_0^\infty e^{-t} dZ_t)$ with $Z = \{Z_t : t \geq 0\}$ the Lévy process in law given by $Z_t = \Lambda((0, t])$. As previously noted this means that $\tilde{\mu} = \Phi(\mathcal{L}(Z_1))$. But since, by (1), $\tilde{\mu} = \Phi^{(k)}(\mu)$ it follows that $\mathcal{L}(Z_1) = \Phi^{(k-1)}(\mu)$, i.e. $\mathcal{L}(\Lambda((0, 1])) = \Phi^{(k-1)}(\mu)$. \hfill $\square$

4 Multiparameter Ornstein-Uhlenbeck processes

For $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$ and $b = (b_1, \ldots, b_k) \in \mathbb{R}^k$ write $a \leq b$ if $a_j \leq b_j$ for all $j$ and $a < b$ if $a_j < b_j$ for all $j$. Define the half-open interval $[a, b]$ as $[a, b] = \{t \in \mathbb{R}^k : a \leq t \leq b\}$ and let $[a, b] = \{t \in \mathbb{R}^k : a < t \leq b\}$. Further, let $\mathcal{A} = \{t \in \mathbb{R}^k : t_j = 0 \text{ for some } j\}$, and for $\mathcal{R} = (R_1, \ldots, R_k)$ where $R_j$ is either $\leq$ or $\geq$ $a \mathcal{R} b$ if $a, R_j, b_j$ for all $j$.

Consider a family $F = \{F_t : t \in S\}$ where $S$ is either $\mathbb{R}^k$ or $\mathbb{R}^k_+$ and $F_t \in \mathcal{F}$ for all $t \in S$. For $a, b \in S$ with $a \leq b$ define the increment of $F$ over $[a, b]$, $\Delta^b_a F$, as

$$\Delta^b_a F = \sum_{\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{0, 1\}^k} (-1)^{c(\epsilon)} F_{(c(\epsilon_1), \ldots, c(\epsilon_k))} ,$$

where $c^j(0) = b^j$ and $c^j(1) = a^j$. That is, $\Delta^b_a F = F_{b} - F_a$ if $k = 1$, and $\Delta^b_a F = F_{(b_1, \ldots, b_k)} + F_{(a_1, a_2)} - F_{(a_1, b_2)} - F_{(b_1, a_2)}$ if $k = 2$. Note that $\Delta^b_a F = 0$ if $a < b$ and $b - a \in \mathcal{A}$. We say that $F = \{F_t : t \in S\}$ is lamp if the following conditions are satisfied:

(i) for $t \in \mathbb{R}^k_+$ the limit $F(t, \mathcal{R}) = \lim_{t \to u^+; t \in \mathcal{U}} F_u$ exists for each of the $2^k$ relations $\mathcal{R} = (R_1, \ldots, R_k)$ where $R_j$ is either $\leq$ or $\geq$. When $S = \mathbb{R}^k_+$ let $F(t, \mathcal{R}) = F_t$ if there is no $u$ with $t \mathcal{R} u$.

(ii) $F_t = F(t, \mathcal{R})$ for $\mathcal{R} = (\leq, \ldots, \leq)$;

Here lamp stands for limits along monotone paths. See Adler et al. [2] for references to the literature on lamp trajectories. When $S = \mathbb{R}^k_+$, one often assumes in addition that $F_t = 0$ for $t \in \mathcal{A}$, i.e. $F$ vanishes on the axes.
Assume that $M = \{ M(A) : A \in \mathfrak{H}(\mathbb{R}^k) \}$ is a homogeneous Lévy basis on $\mathbb{R}^k$ associated with $\mu \in \text{ID}$ which has characteristic triplet $(\sigma^2, \nu, \gamma)$. Define $U = \{ U_t : t \in \mathbb{R}^k_+ \}$ and $X = \{ X_t : t \in \mathbb{R}^k_+ \}$ as

$$ U_t = \int_{[0,t]} e^s \ M(ds) \quad \text{and} \quad X_t = e^{-t} U_t \quad \text{for } t \in \mathbb{R}^k_+. \quad (4.1) $$

Since for $a, b \in \mathbb{R}^k_+$ with $a \leq b$,

$$ \Delta^b_a U = \int_{(a,b)} e^s \ M(ds), $$

the random variables $\Delta^b_a U, \ldots, \Delta^{b_{n+1}} a U$ are independent whenever $(a^1, b^1], \ldots, (a^n, b^n]$ are disjoint intervals in $\mathbb{R}^k_+$. Moreover, $U$ is continuous in probability since $M$ is homogeneous. Thus, $U$ is a Lévy process in the sense of Adler et al. [2], p. 5, and a Lévy sheet in the sense of Dalang and Walsh [4] (in the case $k = 2$). It hence follows e.g. from [2], Proposition 4.1, that by modification we may and do assume that $U$, and hence also $X$, is lamp. Similarly, we may and do assume that $t \mapsto M((0,t])$ is lamp for $t \in \mathbb{R}^k_+.$

Assuming in addition that (3.5) is satisfied we can define processes $V = \{ V_t : t \in \mathbb{R}^k \}$ and $Y = \{ Y_t : t \in \mathbb{R}^k \}$ as

$$ V_t = \int_{s \leq t} e^s \ M(ds) \quad \text{and} \quad Y_t = e^{-t} V_t \quad \text{for } t \in \mathbb{R}^k. \quad (4.2) $$

For fixed $t \in \mathbb{R}^k$ define $\phi^t : \mathbb{R}^k \to \mathbb{R}^k$ as $\phi^t(s) = t - s$. By Lemma 2.1 and the fact that $M$ and $M_{\phi^t}$ are homogeneous Lévy bases associated with $\mu$ we have

$$ Y_t = \int_{s \leq t} e^{-(t-s)} \ M(ds) = \int_{\mathbb{R}^k_+} e^{-s} M_{\phi^t}(ds) = \int_{\mathbb{R}^k_+} e^{-s} M(ds) \quad \text{for } t \in \mathbb{R}^k. $$

That is, $Y_t$ has the same law as the three integrals in (3.1). The same kind of arguments show that $Y$ is stationary in the sense that

$$ (Y_t, \ldots, Y_{t^n}) \overset{d}{=} (Y_{t+t^1}, \ldots, Y_{t+t^n}) \quad \text{for all } n \geq 1 \text{ and } t, t^1, \ldots, t^n \in \mathbb{R}^k. $$

When $k = 1$, $Y$ is often referred to as an Ornstein-Uhlenbeck process. The above is a natural generalization so we shall call $Y$ a $k$-parameter Ornstein-Uhlenbeck process. There are many nice representations and properties of $Y$ as the next remarks illustrate.

Remark 4.1. Denote a generic element in $\mathbb{R}^{k-1}$ by $\tilde{t} = (t_1, \ldots, t_{k-1})$ and let $\tilde{s} = \sum_{j=1}^{k-1} t_j$. A generic element $t$ in $\mathbb{R}^k$ can then be decomposed as $t = (\tilde{t}, t_k)$. For $B \in \mathfrak{B}(\mathbb{R})$ and $\tilde{t} \in \mathbb{R}^{k-1}$, $\{ \tilde{s} \leq \tilde{t} \} \times B$ is the subset of $\mathbb{R}^k$ given by

$$ \{ \tilde{s} \leq \tilde{t} \} \times B = \{ s = (\tilde{s}, s_k) : \tilde{s} \leq \tilde{t} \text{ and } s_k \in B \}. $$

Assuming that (3.5) is satisfied, $Y_t$ is representable as

$$ Y_t = e^{-t_k} \int_{-\infty}^{t_k} e^{\tilde{t}_k} M^k(d\tilde{s}_k) = e^{-\tilde{t}} \int_{\tilde{s} \leq \tilde{t}} e^{\tilde{s}} M^k(d\tilde{s}). \quad (4.3) $$
Hence it suffices to show: Moreover, denoting the variance by \( \text{Var} \), we have
\[
\text{Var}^{\iota} = e^{-\iota} \int_{[0,t] \times B} e^s M(ds), \quad B \in \mathcal{B}^k(\mathbb{R})
\]
\[
\text{Var}^{\varsigma} = e^{-\varsigma} \int_{C \times (\infty, t_\varsigma]} e^s M(ds), \quad C \in \mathcal{B}^{k-1}(\mathbb{R}).
\]

Arguments as in the proof of Theorem 3.1(3) show that \( M^\iota \) and \( M^\varsigma \) are well-defined homogeneous Lévy bases associated with respectively \( \Phi^{(k-1)}(\mu) \) and \( \Phi(\mu) \), and we have (4.3).

The first expression in (4.3) shows that for fixed \( \iota \), \( \{Y_t : t_\iota \in \mathbb{R}\} \) is a one-parameter Ornstein-Uhlenbeck process. By the second expression, \( \{Y_t : t \in \mathbb{R}\} \) is a \((k-1)\)-parameter Ornstein-Uhlenbeck process for fixed \( t_\varsigma \).

**Remark 4.2.** (1) Assume \( \int_{\mathbb{R}} x^2 v(dx) < \infty \). By [12], Corollary 25.8, \( \mu \) has finite second moment. Moreover, denoting the variance by \( \text{Var} \), we have \( \text{Var}(M(A)) = \text{Leb}(A)(\sigma^2 + \int_{\mathbb{R}} x^2 v(dx)) \) for \( A \in \mathcal{B}^k(\mathbb{R}) \). In this case (3.5) is satisfied, implying that \( Y \) is well-defined. The characteristic triplet \((\tilde{\sigma}^2, \tilde{\nu}, \tilde{\gamma}) \) of \( \mathcal{L}(Y_t) \), \( t \in \mathbb{R}^k \), is given in (3.2)–(3.4). Since, by (3.3), \( \int_{\mathbb{R}} x^2 v(dx) = 2^{-k} \int_{\mathbb{R}} x^2 v(dx) < \infty \) it follows that \( Y_t \) is square-integrable with
\[
\text{Var}(Y_t) = \frac{1}{2^k} (\sigma^2 + \int_{\mathbb{R}} x^2 v(dx)).
\]

Let us find the covariance function of \( Y \). Let, for \( j = 1, 2 \), \( t_j = (t_{1j}, \ldots, t_{kj}) \in \mathbb{R}^k \). Set \( t = (t_{11} \wedge t_{21}, \ldots, t_{1k} \wedge t_{2k}) \in \mathbb{R}^k \), \( D^j = \{ s \in \mathbb{R}^k : s \leq t_j \} \) and \( D = \{ s \in \mathbb{R}^k : s \leq t \} = D^1 \cap D^2 \). For \( j = 1, 2 \),
\[
Y_{t_j} = e^{-(t_{1j} - t)} Y_t + e^{-t} \int_{D \setminus D^j} e^s M(ds). \quad (4.4)
\]

Since \( D^1 \setminus D \) and \( D^2 \setminus D \) are disjoint the last term on the right-hand side of (4.4) with \( j = 1 \) is independent of the corresponding term with \( j = 2 \). Thus,
\[
\text{Cov}(Y_{t_j}, Y_{t_2}) = e^{-(t_{1j} - t_2)} e^{-(t_{12} - t)} \text{Var}(Y_{t_1}). \quad (4.5)
\]

(2) Now let \( v \) and \( \gamma \) be zero. In this case Hirsch and Song [5] defined a \( k \)-parameter Ornstein-Uhlenbeck \( \tilde{Y} = \{ \tilde{Y}_t : t \in \mathbb{R}^k \} \) as \( \tilde{Y}_t = e^{-(x_{2t})} M(\{0, e^{2t}\}) \) where \( e^{2t} = (e^{2t_1}, \ldots, e^{2t_k}) \). With \( t, t_1 \) and \( t_2 \) given as under (1) we have
\[
\text{Cov}((\tilde{Y}_{t_1}, \tilde{Y}_{t_2}) = \sigma^2 e^{-(t_{12})} e^{-(t_{21})}. \quad (4.6)
\]

Thus, from (4.5) and Gaussianity it follows that up to a scaling constant \( Y \) and \( \tilde{Y} \) have the same distribution; in other words, in the Gaussian case our definition of a \( k \)-parameter Ornstein-Uhlenbeck process coincides essentially with that of [5].

**Remark 4.3.** Assume (3.5) is satisfied. We may and do assume that \( V_\varsigma \) and thus also \( Y \), is lamp. To see this, note that for arbitrary \( s = (s_1, \ldots, s_k) \in \mathbb{R}^k \) and \( t = (t_1, \ldots, t_k) \in \mathbb{R}_+^k \setminus \mathcal{A} \) we have
\[
V_{s+t} = \Delta^+(s+t) V - \sum_{e \in \{0, 1\}^k, e \neq (1, \ldots, 1)} (-1)^{e} V_{(s_{1} + e_{1}, \ldots, s_{k} + e_{k})}. \quad (4.7)
\]

Hence it suffices to show:
(i) for arbitrary \( s \in \mathbb{R}^k \) the process \( \{ \Delta_{s}^{t+} V : t \in \mathbb{R}^k_{+} \} \) has a lamp modification.

(ii) If at least one coordinate is fixed, then \( V \) has a lamp modification in the remaining coordinates. That is, if e.g. \( t_k = 0 \) then \( \bar{t} = (t_1, \ldots, t_{k-1}) \mapsto V(\bar{t},0) \) is a.s. lamp on \( \mathbb{R}^{k-1} \).

Condition (i) follows as for \( U \) above since \( \Delta_{s}^{t+} V = \int_{(s, t]} e^u M(du) \). To check (ii) consider for simplicity the case where \( t_k \) is fixed at \( t_k = 0 \) while all other coordinates vary freely. As in Remark 4.1 we have

\[
V(\bar{t},0) = \int_{\mathbb{R}^k} e^{\bar{s}} \tilde{M}(d\bar{s}) \quad \text{for} \quad \bar{t} \in \mathbb{R}^{k-1},
\]

where \( \tilde{M} = \{ \tilde{M}(B) : B \in \mathcal{A}_b(\mathbb{R}^{k-1}) \} \) is the homogeneous Lévy basis on \( \mathbb{R}^{k-1} \) given by

\[
\tilde{M}(B) = \int_{B \times (-\infty,0]} e^{s} M(ds), \quad B \in \mathcal{A}_b(\mathbb{R}^{k-1}).
\]

By recursion we can reduce to \( k = 1 \) in which case \( t \mapsto \int_{-\infty}^{t} e^{s} M(ds) \) has a càdlàg modification, implying the result.

From now on let \( k = 2 \). Recall that \( X \) and \( Y \) are defined in (4.1) and (4.2). If \( t = (t_1, t_2) \) write \( X_{t_1,t_2} \) as an alternative to \( X_1 \).

**Proposition 4.4.** With probability one we have for all \( t = (t_1, t_2) \in \mathbb{R}^2_+ \) that

\[
X_{t_1,t_2} = M((0, t]) - \int_{0}^{t_1} X_{s_1,t_2} ds_1 - \int_{0}^{t_2} X_{t_1,s_2} ds_2 - \int_{0}^{t_1} \int_{0}^{t_2} X_{s_1,s_2} ds_1 ds_2.
\]

Assume that (3.5) is satisfied. Then with probability one we have for all \( t = (t_1, t_2) \in \mathbb{R}^2_+ \) that

\[
Y_{t_1,t_2} = Y_{0,t_2} + Y_{t_1,0} - Y_{0,0} + M((0, t]) - \int_{0}^{t_1} Y_{s_1,t_2} ds_1 - \int_{0}^{t_2} Y_{t_1,s_2} ds_2
\]

\[
+ \int_{0}^{t_1} Y_{s_1,0} ds_2 + \int_{0}^{t_2} Y_{0,s_2} ds_2 - \int_{0}^{t_1} \int_{0}^{t_2} Y_{s_1,s_2} ds_1 ds_2.
\]

**Proof.** Since the proofs are similar we only prove the representation of \( Y \).

First we fix \( t_2 \in \mathbb{R}_+ \). Arguing as in the proof of Theorem 3.1(3), cf. also Remark 4.1, we can represent \( \{ V_{t_1,t_2} : t_1 \geq 0 \} \) as

\[
V_{t_1,t_2} = V_{0,t_2} + \int_{0}^{t_1} e^{s_1} U^{t_2}(ds_1),
\]

where \( U^{t_2} = \{ U^{t_2}(B) : B \in \mathcal{A}_b(\mathbb{R}_+) \} \) is the homogeneous Lévy basis given by

\[
U^{t_2}(B) = \int_{B \times (-\infty,t_2]} e^{s_2} M(ds), \quad B \in \mathcal{A}_b(\mathbb{R}_+).
\]

Thus, \( \{ V_{t_1,t_2} : t_1 \geq 0 \} \) is a semimartingale in the filtration of the Lévy process \( \{ U^{t_2}((0, t_1]) : t_1 \geq 0 \} \).

Let \( R = \{ R_t : t \in \mathbb{R}^2_+ \} \) be given by

\[
R_t = \int_{(0,t_1) \times (-\infty,t_2]} e^{s_2} M(ds) = U^{t_2}((0, t_1]) \quad \text{for} \quad t \in \mathbb{R}^2_+.
\]
Since $R$ has independent increments and is continuous in probability we may and do assume that it is lamp, see [2].

For fixed $t_2$, $\{V_{t_1, t_2} : t_1 \leq 0\}$ is a semimartingale we can apply integration by parts, together with Lemma 2.1(1) and the fact that all terms are lamp, to obtain

$$e^{-s_1} V_t = V_{t_1, t} + \int_0^{t_1} e^{-s_1} U^{t_2}(ds_1) - \int_0^{t_1} e^{-s_1} V_{s_1, t_2} ds_1$$

$$= V_{t_1, t} + R_{t_1, t_2} - \int_0^{t_1} e^{-s_1} V_{s_1, t_2} ds_1 \quad \text{for all } t = (t_1, t_2) \in \mathbb{R}_+^2 \quad \text{a.s.} \quad (4.6)$$

The same kind of argument for $t_2$ instead of $t_1$ gives

$$Y_{t_1, t_2} = e^{-t_2} V_{t_1, t_2} + e^{-t_2} R_{t_1, t_2} - \int_0^{t_1} Y_{s_1, t_2} ds_1$$

$$= e^{-t_2} V_{t_1, t_2} + R_{t_1, 0} + M((0, t]) - \int_0^{t_2} e^{-s_2} R_{t_1, s_2} ds_2$$

$$- \int_0^{t_1} Y_{s_1, t_2} ds_1 \quad \text{for all } t = (t_1, t_2) \in \mathbb{R}_+^2 \quad \text{a.s.}$$

From (4.6) we have

$$e^{-t_2} R_{t_1, t_2} = Y_{t_1, t_2} - Y_{0, t_2} + \int_0^{t_1} Y_{s_1, t_2} ds_1 \quad \text{for all } (t_1, t_2) \in \mathbb{R}_+^2 \quad \text{a.s.}$$

and hence

$$\int_0^{t_2} e^{-s_2} R_{t_1, s_2} ds_2 = \int_0^{t_2} Y_{t_1, s_2} ds_2 - \int_0^{t_2} Y_{0, s_2} ds_2$$

$$+ \int_0^{t_1} \int_0^{t_2} Y_{s_1, s_2} ds_2 ds_1 \quad \text{for all } (t_1, t_2) \in \mathbb{R}_+^2 \quad \text{a.s.}$$

Inserting this we get

$$Y_{t_1, t_2} = e^{-t_2} V_{t_1, t_2} + R_{t_1, 0} + M((0, t]) - \int_0^{t_1} Y_{s_1, t_2} ds_1$$

$$- \int_0^{t_2} Y_{t_1, s_2} ds_2 + \int_0^{t_2} Y_{0, s_2} ds_2 - \int_0^{t_1} \int_0^{t_2} Y_{s_1, s_2} ds_2 ds_1$$

$$= Y_{0, t_2} + R_{t_1, 0} + M((0, t]) - \int_0^{t_1} Y_{s_1, t_2} ds_1 - \int_0^{t_2} Y_{t_1, s_2} ds_2$$

$$+ \int_0^{t_2} Y_{0, s_2} ds_2 - \int_0^{t_1} \int_0^{t_2} Y_{s_1, s_2} ds_2 ds_1 \quad \text{for all } t = (t_1, t_2) \in \mathbb{R}_+^2 \quad \text{a.s.}$$

Using (4.6) with $t_2 = 0$ we get

$$R_{t_1, 0} = Y_{t_1, 0} - Y_{0, 0} + \int_0^{t_1} Y_{s_1, 0} ds_1 \quad \text{for all } t_1 \geq 0 \quad \text{a.s.}$$

The result follows by inserting this in the expression for $Y_{t_1, t_2}$. \qed
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References


