A WEAK LAW OF LARGE NUMBERS FOR THE SAMPLE COVARIANCE MATRIX

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Abstract
In this article we consider the sample covariance matrix formed from a sequence of independent and identically distributed random vectors from the generalized domain of attraction of the multivariate normal law. We show that this sample covariance matrix, appropriately normalized by a nonrandom sequence of linear operators, converges in probability to the identity matrix.

1. Introduction:
Let \( X, X_1, X_2 \cdots \) be iid \( R^d \) valued random vectors with \( \mathcal{L}(X) \) full. The condition of fullness is the multivariate analogue of nondegeneracy and will be in force throughout this article. It means that \( \mathcal{L}(X) \) is not concentrated on any \( d - 1 \) dimensional hyperplane. Equivalently, \( \langle X, \theta \rangle \) is nondegenerate for every \( \theta \). Here \( \langle \cdot, \cdot \rangle \) denotes the inner product.
Throughout this article all vectors in \( R^d \) are assumed to be column vectors. For any matrix, \( A \), \( A^t \) denotes its transpose. Let \( \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \). We denote and define the sample covariance matrix by \( C_n = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)(X_i - \bar{X}_n)^t \). That \( C_n \) has a unique nonnegative symmetric square root, denoted above by \( C_{n}^{1/2} \), follows from the fact that \( \langle C_n \theta, \theta \rangle = \sum_{i=1}^n \langle X_i - \bar{X}_n, \theta \rangle^2 \geq 0 \), so that \( C_n \) is nonnegative. Also, \( C_n \) is clearly symmetric. However, there is no guarantee that \( C_n \) is invertible with probability one.
In [3] we describe two ways to circumvent the problem of lack of invertibility of \( C_n \). One such approach is to define
\[
B_n = \begin{cases} 
C_n & \text{if } C_n \text{ is invertible} \\
I & \text{otherwise}
\end{cases}
\] (1.3)
The success of this approach relies on the fact that if \( \mathcal{L}(X) \) is in the Generalized Domain of Attraction of the Normal Law (see (1.6) below for the definition), then \( P(C_n = B_n) \rightarrow 1 \). (See
In light of this, we will assume without loss of generality that $C_n$ is invertible. $\mathcal{L}(X)$ is said to be in the Generalized Domain of Attraction (GDOA) of the Normal Law if there exist matrices $A_n$ and vectors $v_n$ such that
\[ A_n \sum_{i=1}^{n} X_i - v_n \Rightarrow N(0, I). \] 
One construction of $A_n$ is such that $A_n$ is invertible, symmetric and diagonalizable. See Hahn and Klass [2].

The main result is Theorem 1 below. This result was shown in Sepanski [5]. However, there the proof was based on a highly technical comparison of the eigenvalues and eigenvectors of $C_n$ and $A_n$. There the proof was essentially real valued. The purpose of this note is to give a more efficient proof that is operator theoretic and multivariate in nature. For more details, we refer the interested reader to the original article. In particular, Sepanski [5] contains a more complete list of references.

2. Results

**Theorem 1:** If the law of $X$ is in the generalized domain of attraction of the multivariate normal law, then
\[ \sqrt{n} A_n C_n^{1/2} \rightarrow I \quad \text{in pr.} \]

**Proof:** Let $P_n(\omega)$ denote the empirical measure. That is, $P_n(\omega)(A) = \frac{1}{n} \sum_{i=1}^{n} I[X_i(\omega) \in A]$. Here $I$ is the indicator function. For each $\omega \in \Omega$ let $X_1^*, \ldots, X_n^*$ be iid with law $P_n(\omega)$.

Sepanski [4], Theorem 2, shows that under the hypothesis of GDOA,
\[ A_n \sum_{j=1}^{n} X_j^* - n\mu \Rightarrow N(0, I) \quad \text{in pr.} \]

Sepanski [3], Theorem 1, shows that under the hypothesis of GDOA,
\[ (nC_n)^{-1/2} \sum_{j=1}^{n} X_j^* - n\mu \Rightarrow N(0, I) \quad \text{in pr.} \]

These two results, together with the multivariate Convergence of Types theorem of Billingsley [1], imply that
\[ (nC_n)^{-1/2} = B_n R_n A_n, \tag{1} \]
where $B_n \rightarrow I$ in pr., and $R_n$ are (random) orthogonal. The proof of Theorem 1 is thereby reduced to showing that $R_n \rightarrow I$ in pr. However, convergence in probability is equivalent to every subsequence having a further subsequence which converges almost surely. This reduces the proof to a pointwise result about the behavior of the linear operators.

Write $A_n = Q_n D_n Q_n^t$ where $Q_n$ is orthogonal and $D_n$ is diagonal with nonincreasing diagonal entries. Let $P_n = Q_n R_n Q_n^t$ and $K_n = Q_n B_n Q_n^t$.

\[ \|K_n - I\| = \|Q_n^t B_n Q_n - Q_n^t Q_n\| \leq \|B_n - I\| \rightarrow 0 \]

By the same token, $R_n \rightarrow I$ if and only if $P_n \rightarrow I$. Also, $(nC_n)^{-1/2}$ is positive and symmetric and therefore so are $B_n R_n A_n$ and $K_n P_n D_n$. The proof of Theorem 1 is reduced to the following lemma.
Lemma 2: Let $P_n$ be orthogonal. Let $D_n = \text{diag}(\lambda_{n1}, \cdots, \lambda_{nd})$ be diagonal such that $\lambda_{n1} \geq \lambda_{n2} \geq \cdots \geq \lambda_{nd} > 0$. Suppose $K_n \rightarrow I$. If $K_n P_n D_n$ is positive and symmetric for every $n$, then $P_n \rightarrow I$.

Proof: Given a subsequence of $P_n$ we show that there is a further subsequence along which $P_n \rightarrow I$. Let $E_n = \lambda_{n1}^{-1} D_n$. This is a diagonal matrix of all positive entries that are bounded above by 1. Therefore, given any subsequence, there is a further subsequence along which $K_n \rightarrow I$, $P_n \rightarrow P$, and $E_n \rightarrow E$. Necessarily, $P$ is orthogonal and $E$ is diagonal with entries in $[0, 1]$. Furthermore, $E$ has at least one diagonal entry that is 1 and its entries are nonincreasing. Since $K_n P_n E_n$ is symmetric, nonnegative and $K_n \rightarrow I$, we have that $PE = EP^t$, and $PE$ is nonnegative. Now, $(PE)^2 = (PE)^t PE = EP^{-1} PE = E^2$. Hence, since $PE$ and $E$ are both nonnegative, $PE = E$. If $E$ is invertible, then $P = I$ and we are done. Suppose $E$ is not invertible. Write $E = \begin{pmatrix} E(1) & 0 \\ 0 & 0 \end{pmatrix}$ where $E(1)$ is an $m \times m$ invertible diagonal matrix with $m < d$. Next, write $P = \begin{pmatrix} P(1) & P(2) \\ P(3) & P(4) \end{pmatrix}$ where $P(1)$ is an $m \times m$ matrix. Since $PE = E$, we have

$$\begin{pmatrix} P(1) E(1) \\ P(3) E(1) \end{pmatrix} = \begin{pmatrix} E(1) \\ 0 \end{pmatrix}.$$ 

From $P(1) E(1) = E(1)$ and the invertibility of $E(1)$, we have that $P(1) = I_m$. Similarly, from $P(3) E(1) = 0$ we have that $P(3) = 0$. Therefore, $P = \begin{pmatrix} I_m & P(2) \\ 0 & P(4) \end{pmatrix}$. Next, multiplying $PP^t$, and $P^t P$, and equating the $(1, 1)$ entries we have that $I_m + P(2) P^t(2) = I_m$. From this we conclude that $P(2) P^t(2) = 0$, and therefore also, $P(2) = 0$. We have that,

$$P = \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & P(4) \end{pmatrix}.$$

The proof continues inductively. Let $K(n_2), P(n_3), E(n_4)$ be the $(2,2)$ block of $K_n, P_n, E_n$ respectively. $P(n_4)$ may not be orthogonal, but $P(4)$ is. Apply the previous argument to $\begin{pmatrix} K(n_4) P(n_4) P^t(4) \\ P(4) E(n_4) \end{pmatrix}$. Note that $K(n_4) P(n_4) P^t(4) \rightarrow IP(4) P^t(4) = I$, so that we may apply the argument with $K(n_4) P(n_4) P^t(4)$ as the new $K_n$ in the induction step. Since the matrices are all finite dimensional, the argument will eventually terminate.

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REFERENCES


