UNIQUENESS FOR THE SKOROKHOD EQUATION WITH NORMAL REFLECTION IN LIPSCHITZ DOMAINS

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in a domain \( D \), where \( W_t \) is Brownian motion in \( \mathbb{R}^d \), \( \nu \) is the inward pointing normal vector on the boundary of \( D \), and \( L_t \) is the local time on the boundary. The solution to this equation is reflecting Brownian motion in \( D \). In this paper we show that in Lipschitz domains the solution to the Skorokhod equation is unique in law.

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1. Introduction.

We consider the Skorokhod equation in a domain $D \subseteq \mathbb{R}^d$, $d \geq 2$:

\begin{equation}
    dX_t = dW_t + \frac{1}{2} \nu(X_t) dL_t, \quad X_0 = x_0,
\end{equation}

where $W_t$ is $d$-dimensional Brownian motion, $L_t$ is the local time of $X_t$ on the boundary of $D$, and $\nu$ is the inward pointing unit normal vector. It is well-known that in smooth domains $X_t$ is reflecting Brownian motion with normal reflection.

There are various types of solutions to (1.1). Pathwise existence and uniqueness holds for (1.1) when the domain $D$ is a $C^2$ domain. This was proved by Lions and Sznitman [LS]. In fact they considered domains slightly more general than $C^2$, but the class of domains they considered does not contain the class of $C^{1+\alpha}$ domains for any $\alpha \in (0, 1)$. They also considered more general diffusion coefficients and considered oblique reflection as well as normal reflection. Their work was generalized by Dupuis and Ishii [DI], who considered $C^1$ domains, but required the angle of reflection to vary in almost a $C^2$ manner. For normal reflection, this implies the domains must be nearly $C^2$.

Another type of uniqueness is weak uniqueness. That means that there exist processes $X_t$ and $W_t$ satisfying (1.1) where $W_t$ is a Brownian motion, but that $X$ need not be measurable with respect to the $\sigma$-fields generated by $W$. In [BH1] reflecting Brownian motion in bounded Lipschitz domains with normal reflection was constructed using Dirichlet forms, and in [BH2] and [FOT], Ex. 5.2.2, it was shown that this process provides a weak solution to the Skorokhod equation. These results were extended in [FT].

Closely related to weak uniqueness is the submartingale problem of Stroock and Varadhan [SV1]. They proved existence and uniqueness of the submartingale problem corresponding to (1.1) with more general diffusion coefficients and with oblique reflection for $C^2$ domains.

Using Dirichlet forms techniques, Williams and Zheng [WZ] constructed reflecting Brownian motion that provides a weak solution to (1.1) for domains more irregular than Lipschitz domains. Further research along these lines was done by [CFW] and [C]. Uniqueness of reflecting Brownian motion corresponding to the Dirichlet form for Brownian motion can be proved for quite general domains by the techniques of [F].

In this paper we prove weak uniqueness of (1.1) for Lipschitz domains. We prove that there is only one probability measure $\mathbb{P}$ under which $W_t$ is a $d$-dimensional Brownian motion, $X_t$ spends 0 time on the boundary, $L_t$ is the local time of $X_t$ on the boundary (defined as a limit of occupation times), and (1.1) holds. See Theorem 2.2 for a precise statement.

The question of weak uniqueness is a natural one. In problems of weak convergence, (e.g., in proving convergence of penalty methods as in [LS]) one is led to solutions to the Skorokhod equation. If one knew a priori that the solution was associated to a Dirichlet
form, the uniqueness would be easy, but in general one does not know in advance that
the solution corresponds to a Dirichlet form or even that the solution is strong Markov.
Submartingale problems are also a natural class to consider, but in Lipschitz domains
there is considerable difficulty in formulating them; typically, the class of test functions
one would want to consider is empty.

In Section 2 we give definitions and recall a few facts about the reflecting Brownian
motion constructed in [BH1]. We also prove a few preliminary propositions.

The reason problems of weak uniqueness tend to be hard is the paucity of the right
type of functions; this is also the reason problems involving Lipschitz domains are typically
much harder than those involving smoother domains. Section 3 is devoted to constructing
a sequence of functions satisfying certain conditions. An estimate of Dahlberg on harmonic
measure and one of Jerison and Kenig for solutions to the Neumann problem play key roles.

Section 4 contains the proof of weak uniqueness for (1.1) for Lipschitz domains.
The main idea is to show that any two solutions must have the same potentials.

In Section 5 we pose a question about the existence of strong solutions. An affir-
mative answer would imply that in fact pathwise uniqueness holds for (1.1) in Lipschitz
domains. At the present time pathwise uniqueness is not known even for $C^{1+\alpha}$ domains
in the plane.

2. Preliminaries.

Notation. We let $B(x, r)$ denote the open ball of radius $r$ centered at $x$. The letter $c$ with
subscripts will denote constants; we begin renumbering anew at each proposition or theo-
rem. Points $x = (x_1, \ldots, x_d)$ will sometimes be written $(\bar{x}, y)$, where $\bar{x} = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$ and $y = x_d$. We will also use polar coordinates: $x = (r, \theta)$, where $r = |x|$ and
$\theta = x/|x| \in \partial B(0, 1)$. The inner product in $\mathbb{R}^d$ of $x$ and $y$ is written $x \cdot y$.

For a domain $D$ with $x \in \partial D$, the boundary of $D$, we let $\nu(x)$ be the inward pointing
normal vector and $\nu_o(x) = -\nu(x)$ the outward pointing normal vector. We write $\sigma(dx)$ for
surface measure on $\partial D$.

Lipschitz domains. A function $f : \mathbb{R}^{d-1} \to \mathbb{R}$ or $f : \partial B(0, 1) \to \mathbb{R}$ is Lipschitz if there
exists $M$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y$ in the domain of $f$. The smallest
such $M$ is the Lipschitz constant of $f$. A domain $D$ is a Lipschitz domain if for all $z \in \partial D$
there exists a coordinate system $CS_z$, an $r_z > 0$, and a Lipschitz function $\Gamma_z$ such that

$$D \cap B(z, r_z) = \{x = (\bar{x}, y) \in CS_z : y > \Gamma_z(\bar{x})\} \cap B(z, r_z),$$

i.e., locally $D$ looks like the region above the graph of a Lipschitz function. A Lipschitz
domain is star-like (relative to 0) if there exists a Lipschitz function $\varphi : \partial B(0, 1) \to (0, \infty)$
such that $D = \{(r, \theta) : 0 \leq r < \varphi(\theta)\}$.
For each \( z \in \partial D \), where \( D \) is a star-like Lipschitz domain, we let \( V_\beta(z) \) denote the interior of the convex hull of \( \{ z \} \cup B(0, \beta) \). We fix \( \beta \) small enough so that \( V_{2\beta}(z) \subseteq D \) for all \( z \in \partial D \). When we need to emphasize the domain we write \( V^D_\beta(z) \).

\[
U_r = \{ x \in D : \text{dist}(x, \partial D) < r \}.
\]

If \( u \) is a function on \( D \), we let

\[
N(u)(z) = N^D(u)(z) = \sup_{x \in V_\beta(z)} |u(x)|
\]

and

\[
N_r(u)(z) = N^D_r(u)(z) = \sup\{ |u(x)| : x \in V_\beta(z) \cap U_r \}.
\]

\( L^2 \) norms with respect to surface measure on \( \partial D \) will be denoted \( \| f \|_{2, \partial D} \). Thus

\[
\| f \|_{2, \partial D}^2 = \int_{\partial D} |f(z)|^2 \sigma(dz).
\]

We will first prove our results for star-like Lipschitz domains, and then extend them to general Lipschitz domains. Let us describe the special set-up that we first consider.

Let \( D \) be a star-like Lipschitz domain, let \( \rho < (\inf \varphi)/4 \), and let \( K = \overline{B(0, \rho)} \). We will consider open subsets \( G \) of \( \partial B(0,1) \) and we consider the corresponding open subsets \( A = \varphi(G) \) of \( \partial D \):

\[
A = \varphi(G) = \{(r, \theta) : r = \varphi(\theta), \theta \in G \}.
\]

**Reflecting Brownian motion.** In this subsection let us suppose the dimension \( d \) is greater than or equal to 3. Let \( D \) be a Lipschitz domain with \( K \) a compact set contained in \( D \) such that \( K \) has smooth boundary. In [BH1] a strong Markov process \((Q^x, X_t), x \in D\), was constructed that represents reflecting Brownian motion in \( D \) with absorption at \( K \). We recall a few properties and derive some others. See [BH1] for details. Let

\[
T_A = T(A) = \inf\{ t > 0 : X_t \in A \}.
\]

Reflecting Brownian motion in \( D \) has a Green function \( g(x, y) \) that is symmetric in \( x \) and \( y \) for \( x, y \in D - K \), harmonic in \( y \) in \( D - K - \{ x \} \), harmonic in \( x \) in \( D - K - \{ y \} \), vanishes as \( x \) or \( y \) tends to the boundary of \( K \), and there exists \( c_1 \) depending only on \( D \) and \( K \) such that

\[
g(x, y) \leq c_1 |x - y|^{2-d}.
\]
If $D$ is star-like, the constant $c_1$ depends only on $\rho$, $\|\nabla \varphi\|_{\infty}$, inf $\varphi$, and sup $\varphi$. In particular, for each $\rho' > 0$, $g(x, \cdot)$ is bounded in $\overline{D} - K - B(x, \rho')$.

A consequence of (2.3) is that

$$(2.4) \quad \mathbb{E}^x T_K = \int_{\overline{D}-K} g(x, y) \, dy \leq c_2, \quad x \in \overline{D}. $$

Another consequence of (2.3) is that

$$\mathbb{E}^x \int_0^{T_K} 1_{\overline{U}_r}(X_s) \, ds = \int_{\overline{U}_r} g(x, y) \, dy \to 0$$

as $r \to 0$, so $X_t$ spends zero time in $\partial D$, and hence starting at $x \in \partial D$, the process leaves $\partial D$ immediately.

In [BH1] it is proved that there exists a continuous additive functional $L_t$ corresponding to the measure $\sigma(dy)$:

$$\mathbb{E}^x L_{T_K} = \int_{\partial D} g(x, y) \sigma(dy), \quad x \in \overline{D},$$

and $L_t$ increases only when $X_t$ is in the support of $\sigma$, namely $\partial D$. It follows from (2.3) that $\mathbb{E}^x L_{T_K} \leq c_3$, $x \in \overline{D}$, where $c_3$ depends on the domain $D$. When $D$ is star-like, $c_3$ depends on $\rho$, $\|\nabla \varphi\|_{\infty}$, inf $\varphi$, and sup $\varphi$. Suppose $f_m$ are nonnegative bounded functions supported in $D - K$ such that $f_m(y) \, dy$ converges weakly to $\sigma(dy)$ (this is the usual weak convergence of measures in probability theory, except that we do not assume the total mass is one) and also that there exist $c_4 > 0$ and $\gamma \in [0, 1)$ such that

$$(2.5) \quad \int_{B(x, s) \cap \overline{D}} f_m(y) \, dy \leq c_4 (s \wedge 1)^{d-1-\gamma}, \quad x \in D, s > 0. $$

An example of $f_m$ satisfying (2.5) is $f_m(y) = a_m^{-1} 1_{U_{1/m}}(y)$, where $a_m$ is the Lebesgue measure of $U_{1/m}$. If the $f_m$ satisfy (2.5), we have by the proof of [BK], Section 2, that $\int g(x, y) f_m(y) \, dy \to \int g(x, y) \sigma(dy)$ uniformly in $x$. Let

$$(2.6) \quad A_m(t) = \int_0^t f_m(X_s) \, ds.$$ 

By [BK], Section 2,

$$\sup_{t \leq T_K} |A_m(t) - L_t| \to 0$$

in probability as $m \to \infty$.

Suppose $D$ is star-like, $\varphi$ is smooth, $x_0 \in \overline{D}$, and $B = (\overline{D} - K) \cap B(x_0, r)$ for some $r > 0$. Then $h(x) = \mathbb{E}^x f(X_{T(B^c)})$ is harmonic in $D \cap B$ and has normal derivative 0 on
There exist $c_5$ and $\alpha$ depending only on $\rho$, the Lipschitz constant of $\varphi$, the supremum and infimum of $\varphi$, and $r$ such that

$$
|h(x) - h(y)| \leq c_5|x - y|^{\alpha}\|f\|_{\infty}, \quad x, y \in (D - K) \cap B(x_0, r/2).
$$

Reflecting Brownian motion satisfies a tightness estimate similar to that of ordinary Brownian motion. By [BH1] there exist $c_6$ and $c_7$ such that if $x \in D$ and $r > 0$,

$$
P^x(\sup_{s \leq t}|X_s - x| \geq \lambda) \leq c_6 e^{-c_7\lambda^2/t}.
$$

**Lemma 2.1.** Suppose $D$ is star-like and $\varepsilon, \eta > 0$. There exists $\delta$ depending only on $\varepsilon, \eta, \|\nabla \varphi\|_{\infty}, \sup \varphi$, and $\inf \varphi$ such that if $\text{dist}(x, \partial D) < \delta$, then

$$
Q^x(T_{\partial B(x, \eta)} < T_{\partial D}) < \varepsilon.
$$

**Proof.** Let $r = \text{dist}(x, \partial D)$. Since $D$ is Lipschitz, there exists $c_1 > 0$ depending only on the Lipschitz constant of $\varphi$ such that if $s \geq r$, then the surface measure of $\partial B(x, 2s) \cap D^c$ is greater than $c_1$ times the surface measure of $\partial B(x, 2s)$. Since the law of $X_t$ up until time $T_{\partial D}$ is the same as that of standard $d$-dimensional Brownian motion and the distribution of Brownian motion on exiting a ball is uniform on the surface of the ball,

$$
Q^x(T_{\partial B(x, 2r)} < T_{\partial D}) \leq 1 - c_1.
$$

Any point $y$ in $\partial B(x, 2r)$ is a distance $2r$ from $x$ and hence no more than $3r$ from $D^c$. So if $y \in D \cap \partial B(x, 2r)$, the same reasoning tells us

$$
Q^y(T_{\partial B(y, 6r)} < T_{\partial D}) \leq 1 - c_1.
$$

By the strong Markov property and the fact that $B(y, 6r) \subseteq B(x, 8r)$ if $|y - x| = 2r$,

$$
Q^x(T_{\partial B(x, 8r)} < T_{\partial D}) \leq (1 - c_1)^2.
$$

We repeat the argument. A point in $\partial B(x, 8r)$ is a distance no more than $9r$ from $D^c$, and a ball of radius $18r$ about such a point is contained in $B(x, 26r)$, so using the strong Markov property,

$$
Q^x(T_{\partial B(x, 26r)} < T_{\partial D}) \leq (1 - c_1)^3.
$$

We continue by induction and obtain

$$
Q^x(T_{\partial B(x, (3^m - 1)r)} < T_{\partial D}) \leq (1 - c_1)^m.
$$
Now choose $m$ so that $(1 - c_1)^m < \varepsilon$ and then choose $\delta$ so that $(3^m - 1)\delta < \eta$. □

**Skorokhod equation.** We now suppose that $d \geq 2$. In [BH2] and [FOT], Ex. 5.2.2, it was shown that the $(Q^x, X_t)$ constructed in [BH1] satisfy the Skorokhod equation: there exists a $d$-dimensional Brownian motion $W_t$ such that

(2.9) \[ dX_t = dW_t + \frac{1}{2} \nu(X_t) dL_t. \]

We want to show that the solution to (2.9) is unique in law. To be precise, let $D$ be an arbitrary Lipschitz domain. We say that

(2.10) a probability measure $\mathbb{P}$ is a solution to the Skorokhod equation (2.9) starting from $x_0 \in \mathcal{D}$ if

(a) \[ \mathbb{P}(X_0 = x_0) = 1, \]

(b) \[ \int_0^\infty 1_{\partial D}(X_s) ds = 0, \]

(c) there exist nonnegative functions $f_m$ with support in $D$ such that $f_m(y) dy$ converges weakly to $\sigma(dy)$, the $f_m$ satisfy (2.5), and for each $t_0$ we have

\[ \sup_{t \leq t_0} |A_m(t) - L_t)| \to 0 \]

in $\mathbb{P}$-probability as $m \to \infty$, where the $A_m$ are defined by (2.6), and

(d) there exists a continuous process $W_t$ which under $\mathbb{P}$ is a $d$-dimensional Brownian motion with respect to the filtration of $X$ such that for all $t$, $X_t \in \mathcal{D}$ and

\[ X_t - X_0 = W_t + \frac{1}{2} \int_0^t \nu(X_s) dL_s. \]

By our discussion above there exists at least one solution to (2.10), namely $Q^{x_0}$. Saying that $W_t$ is a Brownian motion with respect to the filtration generated by $X_t$ means that $W_t - W_s$ has the same distribution as that of a normal random variable with mean 0 and variance $t - s$ and $W_t - W_s$ is independent of $\sigma(X_r; r \leq s)$ whenever $s < t$.

Our main result is the following.
Theorem 2.2. If $D$ is a Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, then there is exactly one solution to (2.10).

The proof of Theorem 2.2 will take up Sections 3 and 4.

The condition (2.10)(c) is slightly stronger than the one sometimes seen in the literature, namely, that $L_t$ be a nondecreasing continuous process that increases only when $X_t \in \partial D$. Here we are essentially requiring the local time $L_t$ to be an additive functional corresponding to surface measure on the boundary.

We will need the following proposition. Let $\theta_t$ be shift operators so that $X_s \circ \theta_t = X_{s+t}$. By [B], Section I.2, we may always suppose such $\theta_t$ exist.

Proposition 2.3. Let $P$ be a solution to (2.10) started at $x_0 \in D$, let $S$ be a finite stopping time, and let $P_S(\omega, d\omega')$ be a regular conditional probability for the law of $X \circ \theta_S$ under $P[\cdot | F_S]$. Then $P$-almost surely, $P_S$ is a solution to (2.10) started at $X_S(\omega)$.

Proof. The proof is standard. Let $A(\omega) = \{\omega' : X_0(\omega') = X_S(\omega)\}$. Then

$$A(\omega) \circ \theta_S = \{\omega' : X_0 \circ \theta_S(\omega') = X_S(\omega)\} = \{\omega' : X_S(\omega') = X_S(\omega)\}. $$

So

$$P(A(\omega) \circ \theta_S | F_S) = 1_{\{X_S(\omega)\}}(X_S) = 1, \quad \text{a.s.}$$

If $B = \{L_t$ is the uniform limit of the $A_m(t)\}$, then

$$B \circ \theta_S = \{L_{t+S} - L_S$ is the uniform limit of $A_m(t + S) - A_m(S)\},$$

and so $P(B \circ \theta_S | F_S) = 1$, a.s. The proof that the process spends 0 time on the boundary under $P_S$ is similar.

Finally, the law of $[X_t - X_0 - \frac{1}{2} \int_0^t \nu(X_s) \, dL_s] \circ \theta_S$ given $F_S$ is the law of $[X_{t+S} - X_S - \frac{1}{2} \int_S^{S+t} \nu(X_s) \, dL_s]$ given $F_S$. This is a Brownian motion by the strong Markov property of Brownian motion. \hfill \Box

3. Some analytic estimates.

We suppose throughout this section that the dimension $d$ is greater than or equal to 3. We start with an estimate on the normal derivative for a mixed boundary problem. We consider standard reflecting Brownian motion $(\mathbb{Q}^r, X_t)$ in $D$, and we kill this process on hitting $K$. Fix a point $x_0 \in D - K$ and choose $\rho'$ small enough so that $\text{dist}(x_0, \partial(D - K)) > 4\rho'$. 
Proposition 3.1. Suppose $D$ satisfies (2.2) and in addition $\varphi$ is $C^\infty$. Suppose $G$ consists of $m$ components such that if $A = \varphi(G)$, then $\sigma(\overline{A} - A) = 0$. Let $g(\cdot)$ be the Green function for $X_t$ killed on hitting $K \cup A$ with pole at $x_0$. Then $(\partial g/\partial \nu)(y)$ exists at almost every point of $\partial D$ (with respect to $\sigma$) and there exists $c_1$ such that

$$\int_A \left( \frac{\partial g}{\partial \nu}(y) \right)^2 \sigma(dy) \leq c_1.$$  

c_1$ depends on $\rho, \rho'$, sup $\varphi$, inf $\varphi$, and the Lipschitz constant of $\varphi$ but does not otherwise depend on $A$. In particular, $c_1$ does not depend on $m$.

Proof. The function $g$ is harmonic in $D - K - \{x_0\}$. By standard results from PDE on the solution to the Dirichlet problem (see [GT], Sections 6.3 and 6.4), $g$ can be extended to be $C^\infty$ at every point of $A$; this means that every point in $A$ has a neighborhood in whose intersection with $\overline{D}$ the function $g$ is $C^\infty$. By standard results on the solution to the Neumann problem (see [GT], Section 6.7), $g$ has a smooth extension up to the boundary in a neighborhood of each point in $\partial D - \overline{A}$. We make no claims at points in $\overline{A} - A$, but this set has surface measure 0.

Let us make the following assumptions about $G$ and $D$. We will show they can be removed at the end of the proof. First we assume that each of the components of $G$ has a piecewise smooth boundary (considered as a subset of the sphere $\partial B(0,1)$).

Let $M$ be the Lipschitz constant of $\varphi$. Let $\theta_1 = (0, \ldots, 0, -1)$. Choose $r$ small (depending only on $M$, sup $\varphi$, and inf $\varphi$) such that there exists a Lipschitz function $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ whose Lipschitz constant is less than $2M$ and with the property that the intersection of $B(\varphi(\theta_1), r)$ with the region above the graph of $\Gamma$ is the same as $D \cap B(\varphi(\theta_1), r)$. Our second assumption is that $G \subseteq \partial B(0,1) \cap B(\theta_1, r)$.

If $H$ is a $(d-1)$-dimensional hyperplane, let $H^+$ be the half space that contains $(0, y)$ for all $y$ sufficiently large. We want to be able to apply Green’s identities in $D - K - B(x_0, \rho)$ with the function $g$, so to do so, we make the following assumption on $D$ for now:

\begin{equation}
(3.1) \text{For each component } E_i \text{ of } G, \text{ there exists } \varepsilon_i > 0 \text{ and a hyperplane } H_i \text{ such that } \{ \varphi(\theta) \in \partial D : \text{dist} (\theta, E_i) \in (0, \varepsilon_i) \} \text{ is contained in } H_i \text{ and } \varphi(\theta) \text{ lies in } H_i^+ \text{ if dist} (\theta, E_i^c) \in (0, \varepsilon_i). \end{equation}

Consider the domain $C_i = \{(r, \theta) \in D - K : \text{dist} (\theta, \partial E_i) < \varepsilon_i/2 \}$. If we let $C_i^R$ be the reflection of $C_i \cap H_i^+$ across $H_i$ and let $C_i^*$ be the interior of $C_i \cap H_i^+ \cup C_i^R$, then $C_i^*$ has Lipschitz boundaries. By the reflection principle, $g$ may be extended across $H_i$. By Dahlberg’s theorem ([B], Section III.5), $\partial g/\partial \nu$ is in $L^2$ with respect to surface measure on $\partial C_i^*$, from which it is follows that $\partial g/\partial \nu$ is in $L^2$ with respect to surface measure on $\partial D \cap \{ \varphi(\theta) : \text{dist} (\theta, E_i^c) \in (0, \varepsilon_i/2) \}$. 

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We do this for each component, and conclude that we may apply Green’s identities to the function \( g \) in \( D - K - B(x_0, 2\rho') \). We can therefore conclude that \( \partial g/\partial y \) is the density of harmonic measure on \( D - K \) started at \( x_0 \); this may be proved exactly as in [B], pp. 217-218.

Let \( S = \partial D \cap B(\varphi(\theta_1), r) \). Let \( F = D - B(0, 2\rho) - B(x_0, 2\rho') \) and let \( y_0 \in B(x_0, 4\rho') - B(x_0, 2\rho') \). Since \( g \) is bounded and harmonic in \( F \), then \( |\nabla g| \) is bounded on \( \partial B(0, 2\rho) \) and on \( \partial B(x_0, 2\rho') \). By the PDE results mentioned in the first paragraph, \( |\nabla g| \) is also bounded in a neighborhood of points of \( S \). Let \((Q^x, X_t)\) be the reflecting Brownian motion constructed in [BH1] and discussed in Section 2. Since \( \partial g/\partial y \) is a harmonic function, we have by Doob’s optional stopping theorem

\[
(3.2) \quad \frac{\partial g}{\partial y}(y_0) = \mathbb{E}^{y_0}[\frac{\partial g}{\partial y}(X_T(\partial F))].
\]

By the fact that \( \Gamma \) is a Lipschitz curve (with Lipschitz constant \( 2M \)) there exists \( c_2 \) depending only on \( M \) such that the ratio of \( \partial g/\partial \nu \) to \( \partial g/\partial y \) is bounded above by \( c_2 \) for \( y \in A \). We have by the definition of harmonic measure that

\[
(3.3) \quad \int_A \left( \frac{\partial g}{\partial \nu}(z) \right)^2 \sigma(dz) = \mathbb{E}^{x_0}\left[ \frac{\partial g}{\partial \nu}(X_T(\partial F)); X_T(\partial F) \in A \right].
\]

Since \( g \geq 0 \) in \( D \) and \( g = 0 \) in \( A \), then \( \partial g/\partial y \geq 0 \) in \( A \). The function

\[
z \mapsto \mathbb{E}^z\left[ \left( \frac{\partial g}{\partial \nu} \right)_A(X_T(\partial F)) \right]
\]

is harmonic, so by Harnack’s inequality, there exists a \( c_3 \) such that

\[
(3.4) \quad \mathbb{E}^{x_0}\left[ \frac{\partial g}{\partial \nu}(X_T(\partial F)); X_T(\partial F) \in A \right] \leq c_3 \mathbb{E}^{y_0}\left[ \frac{\partial g}{\partial \nu}(X_T(\partial F)); X_T(\partial F) \in A \right].
\]

Combining (3.2)-(3.4),

\[
\int_A \left( \frac{\partial g}{\partial \nu}(z) \right)^2 \sigma(dz) \leq c_3 \mathbb{E}^{y_0}\left[ \frac{\partial g}{\partial \nu}(X_T(\partial F)); X_T(\partial F) \in A \right]
\]

\[
\leq c_2 c_3 \mathbb{E}^{y_0}\left[ \frac{\partial g}{\partial y}(X_T(\partial F)); X_T(\partial F) \in A \right]
\]

\[
= c_4 \left( \frac{\partial g}{\partial y}(y_0) - \mathbb{E}^{y_0}\left[ \frac{\partial g}{\partial y}(X_T(\partial F)); X_T(\partial F) \in \partial B(x_0, 2\rho') \cup \partial B(0, 2\rho) \cup S \right] \right.
\]

\[
- \mathbb{E}^{y_0}\left[ \frac{\partial g}{\partial y}(X_T(\partial F)); X_T(\partial F) \in \partial D - S - A \right].
\]

As we argued above, \( \partial g/\partial y \geq 0 \) a.e. in \( \partial D - S \), while the first two terms on the right are bounded by constants depending only on \( \rho, \rho' \), and \( M \). Therefore

\[
(3.5) \quad \int_A \left( \frac{\partial g}{\partial \nu}(z) \right)^2 \sigma(dz) \leq c_5.
\]
We now show how to eliminate the assumptions made near the beginning of the proof. Suppose that we no longer assume $G \subseteq B(\varphi(\theta_1), r)$. Let $A_0 = \varphi(G \cap B(\theta_1, r))$, let $g$ be the Green function for reflecting Brownian motion killed on hitting $K \cup A$ with pole at $x_0$, and let $g_0$ be the Green function for reflecting Brownian motion killed on hitting $K \cup A_0$ with pole at $x_0$. Clearly

$$Q^{x_0}(X_{T(K \cup A)} \in dy) \leq Q^{x_0}(X_{T(K \cup A_0)} \in dy)$$

for $y \in A_0$, so on the set $A_0$ the density of harmonic measure for reflecting Brownian motion killed on hitting $K \cup A$, which is $\partial g/\partial \nu$, is less than or equal to the density of harmonic measure for reflecting Brownian motion killed on hitting $K \cup A_0$, which is $\partial g_0/\partial \nu$. By this fact and (3.5) applied to $g_0$,

(3.6) $$\int_{A \cap B(\varphi(\theta_1), r)} \left( \frac{\partial g}{\partial \nu}(z) \right)^2 \sigma(dz) \leq c_6.$$  

By a rotation of the coordinate system, (3.6) holds when $\theta_1$ is replaced by any other point of $\partial B(0,1)$. Since $\partial D$ can be covered by finitely many balls of the form $B(\varphi(\theta), r)$ with $\theta \in \partial B(0,1)$, summing gives

(3.7) $$\int_A \left( \frac{\partial g}{\partial \nu}(z) \right)^2 \sigma(dz) \leq c_6.$$  

Recall the $\varphi$ is $C^\infty$. If the components $E_i$ of $G$ are each of the form $Q \cap \partial B(0,1)$, where $Q$ is a cube of side length less than $h$, we can achieve (3.1) by modifying $\varphi$ (and hence $D$) slightly. The smaller $h$ is, the less we need to modify $\varphi$. Furthermore, we can approximate $G$ as closely as we like by the union of such components. We can thus find a sequence of star-like domains $D_m$ given by functions $\varphi_m$ converging to $D$ such that (3.7) holds when $A$ is replaced by $A_m = \varphi_m(G)$ and $g$ is replaced by the Green function for reflecting Brownian motion on $D_m$ and $c_6$ is independent of $m$. By the limit argument of [B], pp. 217-218, we thus get (3.7) without any additional assumptions on $G$. □

**Corollary 3.2.** Let $D, G$ and $A$ be as in Proposition 3.1. Let $H$ be $C^\infty$ with support in $D - K$ and let

$$u(x) = \mathbb{E}^x \int_0^{T_A \wedge T_K} H(X_s) \, ds.$$ 

Then $\partial u/\partial \nu$ exists a.e. on $\partial D$ and there exists $c_1$ depending only on $\rho, \rho', \sup \varphi, \inf \varphi$, and the Lipschitz constant of $\varphi$ such that

$$\int_A \left( \frac{\partial u}{\partial \nu}(y) \right)^2 \sigma(dy) \leq c_1.$$
Proof. The almost everywhere existence of $\partial u/\partial \nu$ follows by the same PDE results as were used in the first paragraph of the proof of Proposition 3.1. If $g(x, y)$ is the Green function for reflecting Brownian motion in $D$ killed on hitting $A \cup K$ with pole at $x$, then $u(y) = \int g(x, y)H(x)\,dx$. Since the support of $H$ is a positive distance from $\partial(D - K)$, the result now follows from Proposition 3.1, Fubini’s theorem, and Fatou’s lemma.

We next need an estimate that is essentially that of [JK], Section 4. Suppose $D$ is a star-like Lipschitz domain: $D = \{(r, \theta) : r < \varphi(\theta)\}$, where $\varphi$ is a positive Lipschitz function. Let us suppose $\varphi$ is also $C^\infty$. Let $\psi(\theta)$ be another $C^\infty$ positive function that is strictly less than $\varphi$ for all $\theta$ and let $E = \{(r, \theta) : r < \psi(\theta)\}$. Let $\delta < \text{dist}(\partial D, \partial E)/4$ and let $E^\delta = \{x : \text{dist}(x, \partial E) < \delta\}$. Recall that $\nu_o$ is the outward pointing unit normal vector and $\nu = -\nu$.

**Proposition 3.3.** Let $D$ and $E$ be as above, suppose $f \in L^2(\partial D)$, and suppose $u$ is harmonic in $(D - E) \cup E^\delta$ with $\partial u/\partial \nu_o = f$ on $\partial D$ and $\int_{\partial D} u(z)\,\sigma(dz) = \int_{\partial E} u(z)\,\sigma(dz)$. There exists $c_1$ depending only on $\delta, \sup \varphi, \inf \varphi$, and the Lipschitz constants of $\varphi$ and $\psi$ such that

$$
\|N(\nabla u)\|_{2, \partial D} \leq c_1\|f\|_{2, \partial D} + c_1 \sup_{E^\delta} |u|.
$$

**Proof.** The proof follows [JK], Section 4, closely. First let us suppose $f$ is smooth. Let $h = \partial u/\partial \nu_o$ on $\partial E$, where by $\nu_o$ on $\partial E$ we mean the outward normal vector with respect to the domain $D - E$. Then clearly $u$ is the solution to the Neumann problem in $D - E$ with boundary functions $f$ on $\partial D$ and $h$ on $\partial E$; by Green’s identity, $\int_{\partial D} f = \int_{\partial E} h$. Hence by [GT], Chapter 6, $u$ is smooth on $\overline{D - E}$. Let $x$ be the vector from the point 0 to the point $x$. If

$$
R(x) = |\nabla u(x)|^2 - 2(x \cdot \nabla u(x))\nabla u(x) - (d - 2)u(x)\nabla u(x),
$$

a calculation shows that $\text{div} R(x) = 0$ in $D - E$ since $u$ is harmonic there. So by the divergence theorem,

$$
(3.8) \quad \int_{\partial D} (R \cdot \nu_o)(z)\,\sigma(dz) = \int_{\partial E} (R \cdot \nu_o)(z)\,\sigma(dz).
$$

Let us let $K = \sup_{x \in E^\delta} |u(x)|$. Since $u$ is harmonic in $E^\delta$, then $\nabla u$ is bounded by $c_2 K$ there, and so the right hand side of (3.8) is bounded by $c_3 K^2$.

Let $a(x) = x - (x \cdot \nu_o(x))\nu_o(x)$, so that

$$
(x \cdot \nabla u) = (a(x) \cdot \nabla u) + (x \cdot \nu_o)\frac{\partial u}{\partial \nu_o},
$$

13
Let $\nabla_t u$ denote the tangential component of $\nabla u$, that is,

$$\nabla_t u(x) = (\nabla u(x) \cdot v_1(x), \ldots, \nabla u(x) \cdot v_{d-1}(x)),$$

where $(v_1(x), \ldots, v_{d-1}(x), \nu(x))$ forms an orthonormal set of vectors at $x \in \partial(D - E)$. Then

$$|\nabla u|^2 = |\nabla_t u|^2 + \left(\frac{\partial u}{\partial \nu_o}\right)^2.$$

Since $\partial u/\partial \nu_o = f$ on $\partial D$, we have

$$\int_{\partial D} |\nabla_t u|^2(x \cdot \nu_o(x))\sigma(dx) \leq \int_{\partial D} f^2(x \cdot \nu_o) + 2 \int_{\partial D} |a \cdot \nabla u| + (d - 2) \int_{\partial D} |uf| + c_3 K^2.$$

The domain is bounded, so $x \cdot \nu_o$ and $a$ are bounded, and because $D$ is star-like, there exists $c_4$ such that $x \cdot \nu_o \geq c_4$ on $\partial D$. Hence

$$\int_{\partial D} |\nabla_t u|^2 \leq c_5\left[\int_{\partial D} f^2 + \int_{\partial D} |\nabla_t u| |f| + \int_{\partial D} |u| |f| + K^2\right].$$

We said that $\int_{\partial D} u = \int_{\partial E} u$ and $|u| \leq K$ on $\partial E$. By the Poincaré inequality [M],

$$\frac{1}{\sigma(\partial D)} \int_{\partial D} u^2 = \frac{1}{\sigma(\partial D)} \int_{\partial D} \left(u - \frac{1}{\sigma(\partial D)} \int_{\partial D} u\right)^2 + \left(\frac{1}{\sigma(\partial D)} \int_{\partial D} u\right)^2 \leq c_6\int_{\partial D} |\nabla_t u|^2 + K^2.$$

If we write $F$ for $\int_{\partial D} f^2$ and $I$ for $\int_{\partial D} |\nabla_t u|^2$, then by the Cauchy-Schwarz inequality we have

$$\int_{\partial D} |uf| \leq c_7(I + K^2)^{1/2} F^{1/2}.$$

Substituting in (3.9),

$$I \leq c_8\left[F + I^{1/2} F^{1/2} + K F^{1/2} + K^2\right].$$

This implies there exists $c_9$ depending only on $c_8$ such that $I \leq c_9[K^2 + F]$. Since $\int_{\partial D} |\nabla u|^2 = I + F$, we get $\|\nabla u\|_{2,\partial D} \leq c_{10}[K^2 + F]$. The result now follows for smooth $f$ since the nontangential maximal function is bounded in $L^2$ norm by the $L^2$ norm of the function on the boundary (see [B], Section III.4, or [JK]). Finally we remove the restriction that $f$ be smooth exactly as in [JK], pp. 39-42. □

Suppose $D$ is a domain satisfying the hypotheses of Proposition 3.1, except that now we only assume $\varphi$ is Lipschitz, not necessarily $C^\infty$. Let $K$ be as above. Let $H$ be a nonnegative $C^\infty$ function with support in $D - K$; let $E$ be a smooth domain whose closure is contained in $D$, which contains the support of $H$, and which is star-like with respect to 0. Let $G$ be an open subset of $\partial B(0, 1)$ consisting of finitely many components such that if $A = \varphi(G)$, then $\sigma(\overline{A} - A) = 0$.  

\[14\]
**Proposition 3.4.** Let \(D, K, E, G, A,\) and \(H\) be as above. There exists a function \(u\) that is nonnegative and bounded, \(-1/2)\Delta u = H\) in \(D - K, \) \(\partial u/\partial \nu\) exists a.e., \(u = 0\) a.e. on \(A, \partial u/\partial \nu_o = 0\) a.e. on \(\partial D - A,\) and \(\|N(\nabla u)\|_{2,\partial D} < \infty.\)

**Proof.** Let \(D_n = \{(r, \theta) : r < \varphi_n(\theta)\}\) be domains that are star-like with respect to 0, where the \(\varphi_n\) are \(C^\infty\) and that decrease to \(D;\) we suppose also that \(\sup_n \|\nabla \varphi_n\|\) is finite.

Let \(A_n = \varphi_n(G).\) Let \((Q_n^x, X_t)\) be standard reflecting Brownian motion in \(D_n,\) let expectation with respect to \(Q_n^x\) be written \(E_n^x,\) and let

\[
u_n(x) = E_n^x \int_0^{T_{A_n} \land T_K} H(X_s) \, ds.
\]

Since the support of \(H\) is a compact subset of \(D\) and \(D\) is open, \(\nu_n\) is harmonic in a neighborhood of \(\partial D_n.\) By the discussion in Section 2, \(\partial u_n/\partial \nu = 0\) a.e. on \((\partial D_n) - A_n.\)

By Corollary 3.2, \(\partial u_n/\partial \nu\) exists a.e. on \(\partial D_n\) with \(L^2(\partial D_n)\) norm not depending on \(n.\) So by Proposition 3.3, \(N(\nabla u_n)\) is in \(L^2(\partial D_n)\) with a norm not depending on \(n.\)

We will show that a subsequence of the \(u_n\) converges to a function \(u\) that satisfies

\[-1/2)\Delta u = H\) in \(D, u\) is nonnegative and bounded, \(u = 0\) a.e. on \(A, \partial u/\partial \nu_o = 0\) a.e. on \(\partial D - A,\) and \(\|N(\nabla u)\|_{2,\partial D} < \infty.\)

Note each \(u_n(x)\) is nonnegative and

\[
\|u_n\| \leq \|H\| \sup_{n,y} E_n^y T_K.
\]

By (2.4), the right hand side is finite. By the strong Markov property,

\[
u_n(x) = E_n^x \int_0^{T(\partial E) \land T_K} H(X_s) \, ds + E_n^x u_n(X_{T(\partial E) \land T_K}).
\]

The second term is harmonic inside \(E - K\) and the first is uniformly smooth inside \(E - K\) by standard results from PDE (see [GT], Section 6.4), since \(E\) is smooth. So the \(u_n\) are equicontinuous inside \(E - K.\) On the other hand, each \(u_n\) is harmonic outside the support of \(H.\) Therefore the \(u_n\) are equicontinuous on compact subsets of \(D\) and are uniformly bounded; hence there exists a subsequence which converges uniformly on compact subsets of \(D,\) say to \(u.\) By relabeling the \(D_n,\) let us suppose that the original sequence \(u_n\) converges.

Observe that if \(\theta \in \partial B(0,1),\) then \(V_D^\beta(\varphi(\theta)) \subseteq V_D^\beta_n(\varphi_n(\theta)),\) so \(N^D(\nabla u_n)(\varphi(\theta)) \leq N^D_n(\nabla u_n)(\varphi_n(\theta)).\) Since \(\nabla u_n\) converges uniformly to \(\nabla u\) in compact subsets of \(D - E,\) it follows easily by Fatou’s lemma (cf. [JK], Section 4) that for \(\delta < \text{dist}(\partial D, \partial E),\) we have \(\|N^D_\delta(\nabla u)\|_{2,\partial D} \leq c_1,\) where \(c_1\) depends only on \(\delta\) and the Lipschitz constant of \(\varphi\) but not on \(G.\) It is easy to see that \(\nabla u\) is bounded in \(D - U_\delta\) since \(H\) is smooth and \(u\) is the uniform limit of the \(u_n\) there, so we have

\[
(3.10) \quad \|N^D(\nabla u)\|_{2,\partial D} \leq c_2.
\]
Since \( u \) is bounded, it has nontangential limits a.e. We show the nontangential limit is 0 a.e. on \( A \). Let \( v_n(x) = Q^x_n(X_t \text{ hits } E \text{ before } A_n) \). This is a harmonic function in \( D_n - E \) which equals 1 on \( E \) and has nontangential limits 0 on \( A_n \). Let \( \varepsilon > 0 \) and \( \theta \in G \). Since \( v_n \) is bounded by 1 and is 0 on \( A_n \), by Lemma 2.1 there exists \( \delta \) such that if \( x \in V^D_{\beta n}(\varphi_n(\theta)) \) and \( |x - \varphi_n(\theta)| < \delta \), then \( v_n(x) < \varepsilon \). So if \( x \in V^D_1(\varphi(\theta)) \subseteq V^D_{\beta n}(\varphi_n(\theta)) \) and \( |x - \varphi(\theta)| < \delta/2 \), then for \( n \) large, \( |x - \varphi_n(\theta)| < \delta \). Since \( u_n \) has nontangential limits 0 on \( A_n \) and is harmonic in \( D_n - E \),

\[
0 \leq u_n(x) \leq \sup_n \|u_n\|_\infty Q^x_n(T_E < T_{A_n}) \leq v_n(x) \sup_n \|u_n\|_\infty.
\]

Hence \( u(x) \leq \varepsilon \sup_n \|u_n\|_\infty \). This shows \( u(x) \to 0 \) as \( x \to \varphi(\theta) \) nontangentially.

Since \( \|N_D(\nabla u)\|_{2,\partial D} \leq c_2 \), then \( \nabla u \) converges nontangentially a.e. and so \( \partial u/\partial \nu_o \) exists a.e. It remains to show that \( \partial u/\partial \nu_o = 0 \) a.e. on \( \partial D - A \). Let \( h(\theta) \) be a smooth function with support in \( G^c \). Let \( f_n(x) = E^x_n(h(X_T(\partial D_n)), f(x) = E^x_n(h(X_T(\partial D))), \) where \((Q^x, X_t)\) is reflecting Brownian motion on \( D \). Note the restriction of \( f_n \) to \( \partial D_n \) is supported on \( A^c_n \) and the restriction of \( f \) to \( \partial D \) is supported on \( A^c \).

Let \((P^x, W_t)\) be standard Brownian motion on \( \mathbb{R}^d \). Up until times \( T(\partial D_n) \) and \( T(\partial D) \), \((Q^x, X_t)\) and \((Q^x, X_t)\), respectively, have the same law as \((P^x, W_t)\). So \( f_n(x) = E^x_n(h(W_T(\partial D_n)), f(x) = E^x_n(h(W_T(\partial D))). \) Since \( T_{\partial D_n} \downarrow T_{\partial D} \) and \( h \) is smooth, it follows that \( f_n \) converges to \( f \) on \( D \). Since \( f_n \) and \( f \) are harmonic, the convergence is uniform on compact subsets of \( D \).

By Green’s first identity, since \( f_n \) and \( f \) are harmonic in \( D_n - E \) and \( D - E \), respectively, and \( \partial u_n/\partial \nu_o = 0 \) on \( (\partial D_n) - A_n \),

\[
\int_{D_n - E} \nabla f_n \cdot \nabla u_n = \int_{\partial D_n} f_n \frac{\partial u_n}{\partial \nu_o} + \int_{\partial E} f_n \frac{\partial u_n}{\partial \nu_o} = \int_{\partial E} f_n \frac{\partial u_n}{\partial \nu_o}
\]

and

\[
\int_{D - E} \nabla f \cdot \nabla u = \int_{\partial D} f \frac{\partial u}{\partial \nu_o} + \int_{\partial E} f \frac{\partial u}{\partial \nu_o}.
\]

(Recall \( \nu_o \) is the outward normal vector for the domains \( D_n - E \) or \( D - E \).)

We will show

\[
(3.11) \int_{D_n - E} \nabla f \cdot \nabla u_n \to \int_{D - E} \nabla f \cdot \nabla u.
\]

Since \( f_n \to f \) and \( u_n \) is harmonic on \( \partial E \) and uniformly bounded in \( n \) in a neighborhood of \( E \), \( \partial u_n/\partial \nu_o \to \partial u/\partial \nu_o \) on \( E \). So if \( (3.11) \) holds, \( \int_{\partial D} f(\partial u/\partial \nu_o) = 0 \). If this holds for all such \( h \), then \( \partial u/\partial \nu_o = 0 \) a.e. on \( A^c \). So it remains to show \((3.11) \).

Recall the definition of \( U_r \) from (2.1). We have \( \nabla f_n \to \nabla f \) uniformly on \( D - U_r \) and \( \nabla u_n \to \nabla u \) uniformly on \( D - U_r \), so

\[
(3.12) \int_{D - U_r} \nabla f_n \cdot \nabla u_n \to \int_{D - U_r} \nabla f \cdot \nabla u.
\]
Since $h$ is smooth, by [JK], Theorem 4.13, there exists $c_3$ independent of $n$ such that

$$\int_{\partial D_n} (N(\nabla f_n)(y))^2 \sigma(dy) \leq c_3$$

and

$$\int_{\partial D} (N(\nabla f)(y))^2 \sigma(dy) \leq c_3.$$

Let $\varepsilon > 0$. If $r < \varphi_n(\theta)$, then $|\nabla f_n(r, \theta)| \leq N(\nabla f_n)(\varphi_n(\theta))$. So $\int_{(D_n-D)\cup U_r} |\nabla f_n|^2$ can be made less than $\varepsilon$ if $r$ is small enough and $n$ is large enough, and similarly $\int_{U_r} |\nabla f|^2 < \varepsilon$ if $r$ is small enough. We have

$$\int_{(D_n-D)\cup U_r} |\nabla f_n \cdot \nabla u_n| \leq \left( \int_{(D_n-D)\cup U_r} |\nabla f_n|^2 \right)^{1/2} \left( \int_{(D_n-D)\cup U_r} |\nabla u_n|^2 \right)^{1/2} \leq c_4 \sup_n \varepsilon \|N(\nabla u_n)\|_{2, \partial D}$$

and

$$\int_{U_r} |\nabla f \cdot \nabla u| \leq \left( \int_{U_r} |\nabla f|^2 \right)^{1/2} \left( \int_{U_r} |\nabla u| \right)^{1/2} \leq c_4 \varepsilon \|N(\nabla u)\|_{2, \partial D}.$$

Combining with (3.12) and using the fact that $\varepsilon$ is arbitrary gives (3.11). \hfill $\Box$

**Proposition 3.5.** Let $D, K, E,$ and $H$ be as in Proposition 3.4. Let $G$ be an arbitrary open set in $\partial B(0,1)$ and $A = \varphi(G)$. Then there exists a function $u(x)$ satisfying the conclusions of Proposition 3.4.

**Proof.** Let $G_m$ be open sets satisfying the hypotheses of Proposition 3.1 and increasing to an open set $G$ and let $A_m = \varphi(G_m)$ and $A = \varphi(G)$. Let $u_m(x)$ be the corresponding functions given by Proposition 3.4. The $u_m$ are uniformly bounded, harmonic in $D - E$, and satisfy $-(1/2)\Delta u_m = H$ in $E - K$. As $m$ increases, $A_m \uparrow A$. Using the notation of the proof of Proposition 3.4, observe that as $m$ increases, $T_{A_m}$ decreases, and so $E_n \int_0^{T_{A_m}} H(X_s) ds$ decreases. It follows that $u_m(x)$ decreases as $m$ increases for each $x$. Let $u(x) = \lim_{m \to \infty} u_m(x)$. By the harmonicity and boundedness of the $u_m$, the convergence is uniform on compact subsets of $D - E$. Therefore $u$ is harmonic in $D - E$, bounded in $D$, 0 on $K$, and $-(1/2)\Delta u = H$ in $E - K$.

Suppose $x \in V_\beta(z)$ for some $z \in A$. Then $z \in A_m$ for some $m$ and given $\varepsilon$, there exists $\delta$ such that if $|x - z| < \delta$, then $0 \leq u_m(x) < \varepsilon$. Therefore $u(x) \leq u_m(x) < \varepsilon$. This shows that $u$ has nontangential limits 0 a.e. on $A$.

By Fatou’s lemma and the corresponding result for the $u_m$, $\|N(\nabla u)\|_{2, \partial D} \leq c_1$. So $\partial u/\partial \nu_o$ exists a.e., and we must show that it is 0 a.e. on $A^c$. Suppose there exists a set $B$ of positive surface measure contained in $A^c$ on which $\partial u/\partial \nu_o > r$ for some $r > 0$. (The
case where \( \partial u/\partial \nu_o \) is negative is treated similarly.) Pick \( f \) smooth on \( \partial D \) such that \( f = 1 \) on \( B \) and \( \int_{\partial D} f(\partial u/\partial \nu_o) > r\sigma(B)/2 \). We also require

\[
\left( \int_{\partial D - B} f^2 \right)^{1/2} < \frac{r\sigma(B)}{4 \sup_m \|N(\nabla u_m)\|_{2, \partial D}}.
\]

We can find such a \( f \) by taking smooth \( f \) decreasing to \( 1 \). Extend \( f \) to \( D \) be defining \( f(x) = E_x f(X_T(\partial D)) \), where \((Q^x, X_t)\) is reflecting Brownian motion on \( D \). Since \( B \subseteq A_c \subseteq A_m \), \( \partial u_m/\partial \nu_o = 0 \) on \( B \) and

\[
\left| \int_{\partial D} f \frac{\partial u_m}{\partial \nu_o} \right| = \left| \int_{\partial D - B} f \frac{\partial u_m}{\partial \nu_o} \right| \leq \left( \int_{\partial D - B} f^2 \right)^{1/2} \left( \int_{\partial D} \left| \frac{\partial u_m}{\partial \nu_o} \right|^2 \right)^{1/2} < r\sigma(B)/4.
\]

Now by Green’s identity on \( D \) and the fact that \( f \) is harmonic in \( D \),

\[
\int_{\partial D} f \frac{\partial u_m}{\partial \nu_o} = \int_D \nabla f \cdot \nabla u_m,
\]

and similarly to Proposition 3.4 but easier, this converges to \( \int_D \nabla f \cdot \nabla u = \int_{\partial D} f(\partial u/\partial \nu_o) \). This implies that

\[
\frac{r\sigma(B)}{4} \geq \lim_m \int_{\partial D} f \frac{\partial u_m}{\partial \nu_o} = \int_{\partial D} f \frac{\partial u}{\partial \nu_o} > \frac{r\sigma(B)}{2},
\]

a contradiction. Therefore there exists no such subset \( B \). \( \square \)

**Corollary 3.6.** Let \( D, K, E, H, G, \) and \( A \) be as in Proposition 3.5. There exist reals \( r_n \uparrow 1 \) and functions \( F_n : D \to \mathbb{R} \) that are \( C^\infty \), the \( F_n \) are nonnegative and uniformly bounded, \(-(1/2)\Delta F_n \to H \) uniformly in \( E - K \), \( F_n = 0 \) on \( B(0, \rho/r_n) \), \( F_n \to 0 \) a.e. on \( A \), \( \partial F_n/\partial \nu_o \to 0 \) a.e. on \( A_c \), and \( \sup_n \|N(\nabla F_n)\|_{2, \partial D} < \infty \).

**Proof.** Let \( u \) be the function constructed in Proposition 3.5, let \( r_n \uparrow 1 \) and let

\[
F_n(x) = u(r_n x), \quad x \in D.
\]

\( \square \)

**4. Uniqueness.**

For now we suppose \( D \) satisfies (2.2) and the dimension \( d \geq 3 \). Let \((Q^x, X_t)\) denote a standard reflecting Brownian motion, let \( x_0 \in D \), and let \( \mathbb{P} \) be a probability measure that is a solution to (2.10). Without loss of generality we may suppose \( x_0 \neq 0 \). Our main goal is to show \( \mathbb{P} = Q^{x_0} \). Let \( \rho < \min(|x_0|, \text{dist}(0, \partial D))/4 \) and define \( K = B(0, \rho) \). Let \( \theta_t \) be the usual shift operators.
The process $L_t$ is a continuous process, so if $M > 0$ and $\xi_1(M) = \inf \{ t : L_t \geq M \}$, then $\xi_1(M) > 0$. Since $L_{t \wedge \xi_1(M)}$ is the uniform limit of $A_m(t \wedge \xi_1(M))$, where $A_m$ is defined by (2.6), it follows that $\xi_2(M) = \inf \{ t : \sup_m A_m(t \wedge \xi_1(M)) \geq 2M \}$ is also strictly positive, a.s. We let $\xi_3 = \xi_3(M) = \xi_1(M) \wedge \xi_2(M) \wedge M$, and observe that $0 < \xi_3(M)$ and $\xi_3(M) \to \infty$ as $M \to \infty$.

We define a new probability measure $\mathbb{P}'$ that agrees with $\mathbb{P}$ up to time $\xi_3$ and agrees with $\mathbb{Q}^{x_0}$ after time $\xi_3$ as follows. If $A \in \mathcal{F}_{\xi_3}$ and $B \in \mathcal{F}_\infty$, let

$$\mathbb{P}'(B \circ \theta_{\xi_3} \cap A) = \mathbb{E}_\mathbb{P}(Q^{X(\xi_3)}(B); A).$$

This determines the probability measure $\mathbb{P}'$ on $\mathcal{F}_\infty$ ([SV2], Chapter 6). $\mathbb{P}'$ is a solution to (2.10) up to time $\xi_3$ since it agrees with $\mathbb{P}$ on $\mathcal{F}_{\xi_3}$. $\mathbb{P}'$ solves (2.10) shifted by an amount $\xi_3$ by the fact that for each $x$, $Q^x$ is a solution to (2.10) starting at $x$. If we show that $\mathbb{P}' = Q^{x_0}$, then $\mathbb{P} = Q^{x_0}$ on $\mathcal{F}_{\xi_3(M)}$, and letting $M \to \infty$, we obtain $\mathbb{P} = Q^{x_0}$.

The reason we work with $\mathbb{P}'$ is the following.

**Proposition 4.1.** (a) $\mathbb{E}_\mathbb{P}' L_{T_K} < \infty$.
(b) $\mathbb{E}_\mathbb{P}' \sup_m A_m(T_K) < \infty$.
(c) $\mathbb{E}_\mathbb{P}' T_K < \infty$.

**Proof.** We have $A_m(T_K) = A_m(\xi_3) + A_m(T_K) \circ \theta_{\xi_3}$ and letting $m \to \infty$ we obtain $L_{T_K} = L_{\xi_3} + L_{T_K} \circ \theta_{\xi_3}$. So by the definition of $\mathbb{P}'$,

$$\mathbb{E}_\mathbb{P}' L_{T_K} = \mathbb{E}_\mathbb{P} L_{\xi_3} + \mathbb{E}_\mathbb{P}' (L_{T_K} \circ \theta_{\xi_3})$$

$$= \mathbb{E}_\mathbb{P} L_{\xi_3} + \mathbb{E}_\mathbb{P}(Q^{X(\xi_3)} L_{T_K}).$$

By the definition of $\xi_3$, the first term is bounded by $M$, while the second term is finite by the discussion following (2.4). The proof of assertion (c) is essentially the same.

Since

$$\sup_m A_m(T_K) \leq \sup_m A_m(\xi_3) + (\sup_m A_m(T_K)) \circ \theta_{\xi_3},$$

the proof of (b) is similar. \qed

We now drop the primes from $\mathbb{P}'$, and without loss of generality we may suppose that $\mathbb{E}_\mathbb{P} L_{T_K} < \infty$, $\mathbb{E}_\mathbb{P} \sup_m A_m(T_K) < \infty$, and $\mathbb{E}_\mathbb{P} T_K < \infty$.

Define a measure $\mu$ on $D - K$ by

$$\mu(B) = \mathbb{E}_\mathbb{P} \int_0^{T_K} 1_B(X_s) ds,$$

the amount of time spent in $B$ before hitting $K$. 

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Proposition 4.2. \( \mu \) is absolutely continuous with respect to Lebesgue measure. If we let \( h \) denote the Radon-Nikodym derivative, we may choose \( h \) to be finite, nonnegative, and harmonic in \( D - K - \{x_0\} \).

Proof. The nonnegativity is clear. Let \( B_0 \subseteq B_1 \subseteq B_2 \) be any three concentric balls contained in \( D - K - \{x_0\} \) such that \( \overline{B}_0 \subseteq B_1, \overline{B}_1 \subseteq B_2, \) and \( \overline{B}_2 \subseteq D - K - \{x_0\} \). If we prove that \( \mu \) restricted to \( B_0 \) has a Radon-Nikodym derivative with respect to Lebesgue measure that may be chosen to be harmonic in \( B_0 \), then since \( B_0 \) is arbitrary we will obtain our result.

Let \( S_1 = \inf\{t : X_t \in B_1\}, \ T_1 = \inf\{t > S_1 : X_t \notin B_2\} \), and for \( i \geq 1 \) let \( S_{i+1} = \inf\{t > T_i : X_t \in B_1\} \) and \( T_{i+1} = \inf\{t > S_{i+1} : X_t \notin B_2\} \). If \( B \subseteq B_0 \), then

\[
E^B_\mathbb{P} \int_0^{T_K} 1_B(X_s) \, ds = \sum_{i=1}^\infty E^B_\mathbb{P} \left[ \int_{S_i}^{T_i} 1_B(X_s) \, ds ; S_i < T_K \right].
\]

Note that under \( \mathbb{P} \), the process \( \int_0^t 1_B(X_s) \, ds \) cannot increase before time \( S_1 \) because \( x_0 \notin B_2 \); since \( K \cap B_2 = \emptyset \), if \( S_i < T_K \), then \( T_i < T_K \).

The law of \( X_s \circ \theta_{S_i} \) under a regular conditional probability for \( E[\cdot | \mathcal{F}_{S_i}] \) is by Proposition 2.3 a solution to (2.10) started at \( X_{S_i} \). Started at \( x \in D - K \), the law of \( X_t \) is the same as that of a standard \( d \)-dimensional Brownian motion up to time \( T_{0_D} \). Therefore \( E^B_\mathbb{P} [\int_{S_i}^{T_i} 1_B(X_s) \, ds | \mathcal{F}_{S_i}] \) is the same as the amount of time Brownian motion started at \( X_{S_i} \) spends in \( B \) up until leaving \( B_2 \). If \( g_{B_2}(x,y) \) is the Green function for standard \( d \)-dimensional Brownian motion killed on exiting \( B_2 \), we have then

\[
E^B_\mathbb{P} \left[ \int_{S_i}^{T_i} 1_B(X_s) \, ds ; \mathcal{F}_{S_i} \right] = \int_B g_{B_2}(X_{S_i},y) \, dy.
\]

The law of \( X \circ \theta_{T_i} \) under a regular conditional probability for \( E^\mathbb{P}[\cdot | \mathcal{F}_{T_i}] \) is a solution to (2.10) started at \( X_{T_i} \). A solution started at \( X_{T_i} \) is a standard Brownian motion up until hitting \( \partial D \). By the support theorem for Brownian motion ([B], p. 59), there exists \( \rho < 1 \) such that

\[
Q^y(T_{B_1} < T_K) \leq \rho, \quad y \in \partial B_2.
\]

Hence

\[
Q^{X(T_i)}(T_{B_1} < T_K) \leq \rho,
\]

or

\[
\mathbb{P}(S_{i+1} < T_K | \mathcal{F}_{T_i}) \leq \rho, \quad \text{a.s.}
\]

From this we deduce

\[
\mathbb{P}(S_{i+1} < T_K) = \mathbb{P}(S_{i+1} < T_K, S_i < T_K) = E^\mathbb{P}[\mathbb{P}(S_{i+1} < T_K | \mathcal{F}_{T_i}); S_i < T_K] \\
\leq \rho \mathbb{P}(S_i < T_K).
\]
By induction, $\mathbb{P}(S_i < T_K) \leq \rho^i$.

Combining with (4.1) and (4.2) and using Fubini’s theorem,

\[
\mathbb{E}_\mathbb{P} \int_0^{T_K} 1_B(X_s) \, ds = \int_B \sum_{i=1}^\infty \mathbb{E}_\mathbb{P}[g_{B_2}(X_{S_i}, y); S_i < T_K] \, dy.
\]

$g_{B_2}(x, y)$ is harmonic in $y \in B_0$ when $x \in \partial B_1$; therefore $\mathbb{E}_\mathbb{P}[g_{B_2}(X_{S_i}, y); S_i < T_K]$ is harmonic. Since $g_{B_2}(x, y)$ is bounded over $x \in \partial B_1, y \in B_0$, then

\[
\sum_{i=i_0}^\infty \mathbb{E}_\mathbb{P}[g_{B_2}(X_{S_i}, y); S_i < T_K] \leq \sum_{i=i_0}^\infty c_1 \mathbb{P}(S_i < T_K) \leq \sum_{i=i_0}^\infty c_1 \rho^i < \infty.
\]

Let

\[
h(y) = \sum_{i=1}^\infty \mathbb{E}_\mathbb{P}[g_{B_2}(X_{S_i}, y); S_i < T_K].
\]

In view of (4.3) the sum converges uniformly over $y \in B_0$ and hence $h$ is finite and harmonic in $B_0$.

Since $h$ is nonnegative and harmonic in $D - K - \{x_0\}$, the nontangential maximal function of $h$ is finite a.e. in a neighborhood of $\partial D$, i.e., for $\varepsilon$ less than $\rho'$,

\[
N_\varepsilon(h)(z) < \infty, \quad \text{for almost every } z \in \partial D.
\]

By (2.3), the Green function for $D - K$ with pole at $x_0$ is bounded in $D - B(x_0, \rho')$, say by $R$. We construct a sawtooth domain

\[
D_0 = \bigcup \{V_\beta(z) : z \in \partial D, N_\varepsilon(h)(z) \leq 3R\}.
\]

$D_0$ is a Lipschitz domain, and since it is contained in $D$, still star-like with respect to 0. Let $A = \partial D_0 - \partial D$.

**Lemma 4.3.** There exists $c_1$ depending only on $\varepsilon$ and $R$ such that

\[
\mathbb{E}_\mathbb{P} \int_0^{T_K \wedge T_A} |\varphi(X_t)| \, dL_t \leq c_1 \int_{\partial D - A} |\varphi(y)| \sigma(dy).
\]

**Proof.** First suppose $\varphi$ is nonnegative and continuous on $\partial D$ and extend $\varphi$ to be nonnegative and continuous in $D$ as well. Since $L_t$ is the uniform limit of the $A_m(t)$,

\[
\int_0^{T_K \wedge T_A \wedge t} \varphi(X_s) \, dA_m(s) \to \int_0^{T_K \wedge T_A \wedge t} \varphi(X_s) \, dL_s.
\]
By Proposition 4.1(b) and dominated convergence,

\[(4.4) \quad \mathbb{E}_p \int_0^{T_K \wedge T_A} \varphi(X_s) \, dL_s = \lim_{m \to \infty} \mathbb{E}_p \int_0^{T_K \wedge T_A} \varphi(X_s) \, dA_m(s).\]

\(h\) is bounded by 3R in a neighborhood of \(\partial D_0\). Recall the definition of \(U_r\) from (2.1). Because \(h\) is harmonic in \(D - K - B(x_0, r') - U_r\), it is bounded by a constant \(c_2\) there. Since \(dA_m(s) = f_m(X_s) \, ds\), the right hand side in (4.4) is bounded by

\[\mathbb{E}_p \int_0^{T_K \wedge T_A} \varphi(X_s) f_m(X_s) \, ds \leq \int 1_{D_0}(y) \varphi(y) f_m(y) h(y) \, dy \leq c_3 \int_D \varphi(y) f_m(y) \, dy,\]

where \(c_3 = c_2 \vee R\). Since \(f_m(y) \, dy\) converges weakly to \(\sigma(dy)\) and \(\varphi\) is continuous,

\[\mathbb{E}_p \int_0^{T_K \wedge T_A} \varphi(X_s) \, dL_s \leq c_3 \int_{\partial D} \varphi(y) \sigma(dy).

Now let \(t \to \infty\). By linearity and a limit argument,

\[\mathbb{E}_p \int_0^{T_K \wedge T_A} |\varphi(X_s)| \, dL_s \leq c_3 \int_{\partial D} |\varphi(y)| \sigma(dy),\]

for all \(\varphi\) bounded and measurable on \(\partial D\). Finally, since \(L_t\) grows only when \(X_t \in \partial D\),

\[\mathbb{E}_p \int_0^{T_K \wedge T_A} |\varphi(X_s)| \, dL_s = \mathbb{E}_p \int_0^{T_K \wedge T_A} |(\varphi 1_{\partial D - A})(X_s)| \, dL_s \leq c_3 \int |(\varphi 1_{\partial D - A})(y)| \sigma(dy).

This completes the proof. \(\Box\)

Let \(H\) be a \(C^\infty\) function with support in \(D - K - \{x_0\}\). The key proposition is the following.

**Proposition 4.4.** Let \(u(x) = \mathbb{E}^x \int_0^{T_K \wedge T_A} H(X_s) \, ds\). Then if \(x_0 \in D - K\),

\[\mathbb{E}_p \int_0^{T_K \wedge T_A} H(X_s) \, ds = u(x_0).

**Proof.** Let \(E\) be star-like, contained in \(D\), and containing the support of \(H\). Construct the \(F_n\) as in Corollary 3.6. For \(r > 1\) let \(K_r = B(0, r\rho)\). \(F_n\) is \(C^2\) on \(D - K_r\), so by Ito’s formula,

\[(4.5) \quad F_n(X(T_{K_r} \wedge T_A \wedge t)) - F_n(X_0) = \text{martingale} + \frac{1}{2} \int_0^{T_{K_r} \wedge T_A \wedge t} \Delta F_n(X_s) \, ds + \int_0^{T_{K_r} \wedge T_A \wedge t} \frac{\partial F_n}{\partial \nu}(X_s) \, dL_s,\]

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Take expectations with respect to $\mathbb{P}$. By Lemma 4.3, the expectation of the local time term in (4.5) (i.e., the last term on the right) is bounded by

$$c_1 \int_{\partial D - A} \left| \frac{\partial F_n}{\partial \nu}(y) \right| \sigma(dy).$$

$F_n$ is bounded and $\Delta F_n$ is bounded in $D$, so letting $t \to \infty$, we obtain

$$\mathbb{E}_\mathbb{P} F_n(X(T_{K_r} \wedge T_A)) - F_n(x_0) = \frac{1}{2} \mathbb{E}_\mathbb{P} \int_0^{T_{K_r} \wedge T_A} \Delta F_n(X_s) \, ds + R,$$

where

$$|R| \leq c_1 \int_{\partial D - A} \left| \frac{\partial F_n}{\partial \nu}(y) \right| \sigma(dy).$$

Next let $n \to \infty$. By Corollary 3.6 $\partial F_n / \partial \nu$ is in $L^2(\sigma)$ with a bound independent of $n$ and $\partial F_n / \partial \nu \to 0$ a.e. on $\partial D - A$. We also have that $(1/2)\Delta F_n \to -H$ uniformly on $D - K$, and $F_n \to u$ uniformly. So we obtain

$$\mathbb{E}_\mathbb{P} u(X(T_{K_r} \wedge T_A)) - u(x_0) = -\mathbb{E}_\mathbb{P} \int_0^{T_{K_r} \wedge T_A} H(X_s) \, ds.$$

The function $u$ is 0 on $A$ and on $K$ and $H$ is bounded. So by dominated convergence on the left and monotone convergence on the right, letting $r \downarrow 1$,

$$-u(x_0) = -\mathbb{E}_\mathbb{P} \int_0^{T_{K_r} \wedge T_A} H(X_s) \, ds. \quad \square$$

**Corollary 4.5.** If $x_0 \in D - K$,

$$\mathbb{E}_\mathbb{P} \int_0^{T r \wedge T K} H(X_s) \, ds = \mathbb{E}_{x_0}^{x_0} \int_0^{T r \wedge T K} H(X_s) \, ds$$

for all bounded functions $H$.

**Proof.** This follows from Proposition 4.4 by a limit argument and the fact that the quantity $\mathbb{E}_\mathbb{P} \int_0^{T r \wedge T K} 1_{\{x_0\}}(X_s) \, ds$ is 0 since $X_t$ behaves like a Brownian motion in a neighborhood of $x_0$. \quad \square

**Corollary 4.6.** If $x_0 \in D - K$,

$$\mathbb{E}_\mathbb{P} \int_0^{T K} H(X_s) \, ds = \mathbb{E}_{x_0}^{x_0} \int_0^{T K} H(X_s) \, ds$$

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for all bounded functions $H$.

**Proof.** Recall the definition of $R$ following the proof of Proposition 4.2. If $H$ has support in $D - K - B(x_0, \rho')$, then by the definition of $R$,

$$
\mathbb{E}^{x_0} \int_0^{T_K} H(X_s)1_B(X_s) \, ds \leq R\|H\|_{\infty} |B|
$$

for $B \subseteq D$. This and Corollary 4.5 imply that $h \leq R$, a.e., on $D - B(x_0, \rho')$. Since $h$ is harmonic, it is continuous, or $h \leq R$ on $D - B(x_0, \rho')$. This implies $N_\varepsilon(h)(z) \leq R$, so $D_0 = D$ and hence $A = \emptyset$. □

We would like the conclusion of Corollary 4.6 to hold for all $x_0$, even for $x_0 \in \partial D$.

**Proposition 4.7.** If $x_0 \in \partial D$,

$$
\mathbb{E}_p \int_0^{T_K} H(X_s) \, ds = \mathbb{E}^{x_0} \int_0^{T_K} H(X_s) \, ds.
$$

**Proof.** Let $x_0 \in \partial D$. Let $\xi_4(n) = \inf\{t : |X_t - x_0| \geq 1/n\}$ and $\xi_5(m) = \inf\{t : \text{dist}(X_t, \partial D) \geq 1/m\}$. Choose $m_n$ such that $\mathbb{P}(\xi_5(m_n) > \xi_4(n)) < 1/n$; this is possible since starting at $x_0$ the process under $\mathbb{P}$ leaves $\partial D$ immediately. Let $\xi_6(n) = \xi_4(n) \wedge \xi_5(m_n) \wedge 1/n$. As in Corollary 4.5 it suffices to prove the proposition for $H$ in $C^\infty$ with support in $D - K$. So for $n$ sufficiently large, $X_t$ will not be in the support of $H$ when $t \leq \xi_6(n)$ and

$$
\mathbb{E}_p \int_0^{T_K} H(X_s) \, ds = \mathbb{E}_p \int_{\xi_6(n)}^{T_K} H(X_s) \, ds.
$$

The law of the process $X_s \circ \theta_{\xi_6(n)}$ under a regular conditional probability for $\mathbb{E}_p[\cdot | \mathcal{F}_{\xi_6(n)}]$ is a solution to (2.10) started at $X_{\xi_6(n)}$. On the set where $X_{\xi_6(n)} \notin \partial D$, by Corollary 4.6 we have $\mathbb{E}_p[\int_0^{T_K} H(X_s) \, ds | \mathcal{F}_{\xi_6(n)}] = u(X_{\xi_6(n)})$, where

(4.6) \[ u(x) = \mathbb{E}^x \int_0^{T_K} H(X_s) \, ds. \]

So

$$
\left| \mathbb{E}_p \int_{\xi_6(n)}^{T_K} H(X_s) \, ds - \mathbb{E}_p u(X_{\xi_6(n)}) \right| \leq (\|H\|_\infty \mathbb{E}_p T_K + \|u\|_\infty) \mathbb{P}(X_{\xi_6(n)} \in \partial D)
$$

$$
\leq c_1/n.
$$

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By (2.7), \( u \) is continuous in \( B(x_0, 1/n) \cap \overline{D} \), so letting \( n \to \infty \), we have \( u(X_{\xi_0(n)}) \to u(x_0) \), and hence
\[
\mathbb{E}_\mathbb{P} \int_0^{T_K} H(X_s) \, ds = \mathbb{E}_\mathbb{P} u(x_0) = u(x_0).
\]

\[\Box\]

**Proposition 4.8.** Let \( S_\lambda H = \mathbb{E}_\mathbb{P} \int_0^{T_K} e^{-\lambda t} H(X_t) \, dt \). Then for all \( x_0 \in \overline{D} - K \) and for all \( \lambda < 1/(2 \sup_y \mathbb{E}^y T_K) \),
\[
S_\lambda H = \mathbb{E}^{x_0} \int_0^{T_K} e^{-\lambda t} H(X_t) \, dt.
\]

**Proof.** It is enough to consider \( H \) that are \( C^\infty \) with support in \( D - K \). Let us kill the process on hitting \( K \). Since \( H \) is 0 there, we can let the integrals run from 0 to \( \infty \). Let \( u \) be defined by (4.6). Under a regular conditional probability for \( \mathcal{F}_t \), the law of the process \( X_s \circ \theta_t \) is a solution to (2.10) started at \( X_t \). Therefore by Proposition 4.7
\[
\mathbb{E}_\mathbb{P} \left[ \int_0^\infty H(X_s \circ \theta_t) \, ds \mid \mathcal{F}_t \right] = \mathbb{E}^{X_t} \int_0^\infty H(X_s) \, ds = u(X_t).
\]

We then have
\[
(4.7) \quad S_\lambda u = \mathbb{E}_\mathbb{P} \int_0^\infty e^{-\lambda t} u(X_t) \, dt
\]
\[
= \mathbb{E}_\mathbb{P} \int_0^\infty e^{-\lambda t} \mathbb{E}_\mathbb{P} \left[ \int_0^\infty H(X_{s+t}) \, ds \mid \mathcal{F}_t \right] dt
\]
\[
= \mathbb{E}_\mathbb{P} \int_0^\infty e^{-\lambda t} \int_t^\infty H(X_s) \, ds \, dt
\]
\[
= \mathbb{E}_\mathbb{P} \int_0^\infty H(X_s) \int_0^s e^{-\lambda t} \, dt \, ds
\]
\[
= \mathbb{E}_\mathbb{P} \int_0^\infty H(X_s) \frac{1 - e^{-\lambda s}}{\lambda} \, ds
\]
\[
= \frac{1}{\lambda} u(x_0) - \frac{1}{\lambda} S_\lambda H,
\]

or \( S_\lambda H = u(x_0) - \lambda S_\lambda u \). Define the operator \( R_\lambda \) by
\[
(4.8) \quad R_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t) \, dt.
\]

We thus have \( u = R_0 H \) and so
\[
(4.9) \quad S_\lambda H = R_0 H(x_0) - \lambda S_\lambda R_0 H.
\]

Let
\[
\Theta = \sup_{\|H\|_{\infty} \leq 1} |S_\lambda H - R_\lambda H(x_0)|.
\]

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Note \( \|R_\lambda H\|_\infty \leq \lambda^{-1}\|H\|_\infty \) and \( \|R_0 H\|_\infty \leq c_1\|H\|_\infty \), where \( c_1 = \sup_y E^y T_K \). From the semigroup property of \( Q^x \) (cf. [B], p. 19),

\[
(4.10) \quad R_\lambda H(x_0) = R_0 H(x_0) - \lambda R_\lambda R_0 H(x_0).
\]

Subtracting (4.10) from (4.9),

\[
|S_\lambda H - R_\lambda H(x_0)| = |\lambda(S_\lambda R_0 H - R_\lambda R_0 H(x_0))| \leq \lambda \Theta \|R_0 H\|_\infty \leq \lambda \Theta c_1\|H\|_\infty.
\]

Taking the supremum over \( H \) with \( \|H\|_\infty \leq 1 \), if \( \lambda \leq 1/2c_1 \),

\[
\Theta \leq \lambda \Theta c_1 \leq \Theta/2.
\]

Since

\[
\Theta \leq \sup \left( \frac{|S_\lambda H| + \|R_\lambda H\|_\infty}{\|H\|_\infty} \right) \leq 2/\lambda < \infty,
\]

we have \( \Theta = 0 \) or \( S_\lambda H = R_\lambda H(x_0) \).

Proof of Theorem 2.2. First suppose that \( d \geq 3 \) and \( D \) satisfies (2.2). Recall the notation of Proposition 4.8 and that \( S_\lambda H = R_\lambda H(x_0) \). By the uniqueness of the Laplace transform and the continuity of \( H(X_t) \) when \( H \) is continuous, \( E^\theta H(X_{t\wedge T_K}) = E^{x_0} H(X_{t\wedge T_K}) \). As \( \rho \to 0 \), then \( T_K \to T_{\{x_0\}} \). Since \( X_t \) behaves like a Brownian motion up until time \( T(\partial D) \), then \( T\{x_0\} \) is identically infinite. So \( E^\theta H(X_t) = E^{x_0} H(X_t) \). By standard arguments (see [SV2], Chapter 6), this implies that the finite dimensional distributions of \( X_t \) under \( \mathbb{P} \) and under \( Q^{x_0} \) agree. Therefore \( \mathbb{P} = Q^{x_0} \).

Now let \( D \) be an arbitrary Lipschitz domain. By standard piecing-together arguments (see [SV2]) and (2.8), it suffices to show that for each \( x_0 \in \partial D \), any two solutions \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \) agree in a neighborhood of \( x_0 \). That is, if \( x_0 \in \partial D \), there exists \( r > 0 \) such that \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \) agree on \( \mathcal{F}_{T(\partial B(x_0,r))} \). Inside \( D \), \( X_t \) under both \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \) behaves like ordinary Brownian motion, so we need only consider \( x_0 \in \partial D \). Let a coordinate system and a domain \( D' \) satisfying (2.2) be chosen so that \( D' \) agrees with \( D \) in a neighborhood \( B(x_0,r) \cap D \) of \( x_0 \). Define \( \mathbb{P}'_i \) for \( i = 1, 2 \) by

\[
\mathbb{P}'_i(B \circ \theta_{T(\partial B(x_0,r))} \cap A) = E_{\mathbb{P}_i}(Q_{D'}^{X_{T(\partial B(x_0,r))}}(B); A), \quad A \in \mathcal{F}_{T(\partial B(x_0,r))}, \quad B \in \mathcal{F}_\infty,
\]

where here \( (Q_{D'}^{x}, X_t) \) is the law of reflecting Brownian motion in \( D' \) started at \( x \). As in the discussion preceding Proposition 4.1, \( \mathbb{P}'_i \) is a solution to (2.10) in \( D' \) for \( i = 1, 2 \). By the uniqueness result for domains satisfying (2.2), \( \mathbb{P}'_1 = \mathbb{P}'_2 = Q_{D'}^{x_0} \). So if \( A \in \mathcal{F}_{T(\partial B(x_0,r))} \), then \( \mathbb{P}_1(A) = \mathbb{P}'_1(A) = \mathbb{P}'_2(A) = \mathbb{P}_2(A) \).

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Finally we consider the case of $d = 2$. Let $W_t$ be a one-dimensional Brownian motion reflecting at $-1$ and $1$ and independent of $X_t$. Then the law of $(X_t, W_t)$ is a solution to (2.10) for the Lipschitz domain $D \times (-1, 1)$, and so is unique. The uniqueness of the law of $X_t$ follows easily.

\[ \square \]

5. Strong solutions.

A strong solution to (2.9) exists if there exists a process $X_t$ satisfying (2.9) such that $X$ is measurable with respect to the $\sigma$-fields of $W$. An interesting open problem is the following.

**Problem 5.1.** Does there exist a strong solution to (2.9)?

The reason for our interest is that if a strong solution exists, then in fact pathwise uniqueness holds for (2.9). That is, any two solutions to (2.9) must be identical. The proof of this is simple; cf. [K], Lemma 2.1.

**Proposition 5.2.** Suppose a strong solution to (2.9) exists satisfying (2.10)(a)-(c). Then any two solutions to (2.9) that satisfy (2.10)(a)-(c) agree pathwise, a.s.

**Proof.** Suppose $dY_t = dW_t + (1/2)\nu(Y_t) \, dL_t$ and $Y_t$ is a strong solution. Then there exists a measurable map $F$ from $C[0, \infty)$ to $C[0, \infty)$ such that $Y = F(W)$. Let $X_t$ be another solution. We have

\begin{equation}
W_t = Y_t - \frac{1}{2} \int_0^t \nu(Y_s) \, dL_s, \quad W_t = X_t - \frac{1}{2} \int_0^t \nu(X_s) \, dL_s.
\end{equation}

The uniqueness in law (Theorem 2.2) says that the law of $Y$ is equal to the law of $X$, so using (5.1) the law of the pair $(Y, W)$ is equal to the law of the pair $(X, W)$. Since $Y = F(W)$, then $X = F(W)$, a.s., and we then conclude that $X = F(W) = Y$, a.s. \[ \square \]

**Remark.** We do not know the answer to Problem 5.1 even when $D$ is a $C^{1+\alpha}$ domain and even when the dimension $d$ is 2. (The obvious conformal mapping argument does not appear to help). A $C^{1+\alpha}$ domain is defined analogously to a Lipschitz domain, where we replace Lipschitz functions in the definition by functions whose gradient is Hölder continuous of order $\alpha$. 

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References.


