A PROOF OF A CONJECTURE OF BOBKOV AND HOUDRE

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Abstract
S.G. Bobkov and C. Houdré recently posed the following question on the Internet ([1]): Let \( X, Y \) be symmetric i.i.d. random variables such that:

\[
P\left( \frac{|X + Y|}{\sqrt{2}} \geq t \right) \leq P\{|X| \geq t\},
\]

for each \( t > 0 \). Does it follow that \( X \) has finite second moment (which then easily implies that \( X \) is Gaussian)? In this note we give an affirmative answer to this problem and present a proof. Using a different method K. Oleszkiewicz has found another proof of this conjecture, as well as further related results.

We prove the following:

Theorem. Let \( X, Y \) be symmetric i.i.d. random variables. If, for each \( t > 0 \),

\[
P\{|X + Y| \geq \sqrt{2}t\} \leq P\{|X| \geq t\},
\]

then \( X \) is Gaussian.

Proof. Step 1. \( \mathbb{E}\{|X|^p\} < \infty \) for \( 0 \leq p < 2 \).
For this purpose it will suffice to show that, for \( p < 2 \), \( X \) has finite weak \( p \)'th moment, i.e., that there are constants \( C_p \) such that

\[
P\{|X| \geq t\} \leq C_p t^{-p}.
\]

To do so, it is enough to show that, for \( \epsilon > 0, \delta > 0 \), we can find \( t_0 \) such that, for \( t \geq t_0 \), we have
\[ \mathbb{P}\{|X| \geq (\sqrt{2} + \epsilon)t\} \leq \frac{1}{2 - \delta} \mathbb{P}\{|X| \geq t\}. \tag{2} \]

Fix \( \epsilon > 0 \). Then:

\[
\mathbb{P}\{|X + Y| \geq \sqrt{2}t\} = 2\mathbb{P}\{X + Y \geq \sqrt{2}t\} \\
\geq 2\mathbb{P}\{X \geq (\sqrt{2} + \epsilon)t, Y \geq -ct, \text{ or } Y \geq (\sqrt{2} + \epsilon)t, X \geq -ct\} \\
= 2(2\mathbb{P}\{X \geq (\sqrt{2} + \epsilon)t\} \mathbb{P}\{Y \geq -ct\} - \mathbb{P}\{X \geq (\sqrt{2} + \epsilon)t\}) \\
\geq (2 - \delta)\mathbb{P}\{|X| \geq (\sqrt{2} + \epsilon)t\},
\]

where \( \delta > 0 \) may be taken arbitrarily small for \( t \) large enough. Using (1) we obtain inequality (2).

**Step 2.** Let \( \alpha_1, \ldots, \alpha_n \) be real numbers such that \( \alpha_1^2 + \ldots + \alpha_n^2 \leq 1 \) and let \( (X_i)_{i=1}^{\infty} \) be i.i.d. copies of \( X \); then

\[ \mathbb{E}\{|\alpha_1X_1 + \ldots + \alpha_nX_n|\} \leq \sqrt{2}\mathbb{E}\{|X|\}. \]

We shall repeatedly use the following result:

**Fact:** Let \( S \) and \( T \) be symmetric random variables such that \( \mathbb{P}\{|S| \geq t\} \leq \mathbb{P}\{|T| \geq t\} \), for all \( t > 0 \), and let the random variable \( X \) be independent of \( S \) and \( T \). Then

\[ \mathbb{E}\{|S + X|\} \leq \mathbb{E}\{|T + X|\}. \]

Indeed, for fixed \( x \in \mathbb{R} \), the function \( h(s) = \frac{|s+x|+|s-x|}{2} \) is symmetric and non-decreasing in \( s \in \mathbb{R}_+ \) and therefore

\[ \mathbb{E}\{|S + x|\} = \mathbb{E}\{\frac{|S + x| + |S - x|}{2}\} \leq \mathbb{E}\{\frac{|T + x| + |T - x|}{2}\} = \mathbb{E}\{|T + x|\}. \]

Now take a sequence \( \beta_1, \ldots, \beta_n \in \{2^{-k/2} : k \in \mathbb{N}_0\} \), such that \( \alpha_i \leq \beta_i < \sqrt{2}\alpha_i \). Then \( \beta_1^2 + \ldots + \beta_n^2 \leq 2 \) and

\[ \mathbb{E}\{|\alpha_1X_1 + \ldots + \alpha_nX_n|\} \leq \mathbb{E}\{|\beta_1X_1 + \ldots + \beta_nX_n|\}. \]

If there is \( i \neq j \) with \( \beta_i = \beta_j \) we may replace \( \beta_1, \ldots, \beta_n \) by \( \gamma_1, \ldots, \gamma_{n-1} \) with \( \sum_{i=1}^{n-1} \beta_i^2 = \sum_{j=1}^{n-1} \gamma_j^2 \) and

\[ \mathbb{E}\{|\sum_{i=1}^{n} \beta_iX_i|\} \leq \mathbb{E}\{|\sum_{j=1}^{n-1} \gamma_jX_j|\}. \tag{3} \]

Indeed, supposing without loss of generality that \( i = n - 1 \) and \( j = n \) we let \( \gamma_i = \beta_i \), for \( i = 1, \ldots, n - 2 \) and \( \gamma_{n-1} = \sqrt{2}\beta_{n-1} = \sqrt{2}\beta_n \). With this definition we obtain (3) from (1) and the above mentioned fact.

Applying the above argument a finite number of times we end up with \( 1 \leq m \leq n \) and numbers \( (\gamma_j)_{j=1}^{n} \) in \( \{2^{-k/2} : k \in \mathbb{N}_0\} \), \( \gamma_i \neq \gamma_j \), for \( i \neq j \), satisfying \( \sum_{j=1}^{m} \gamma_j^2 \leq 2 \) and

\[ \mathbb{E}\{|\sum_{i=1}^{n} \alpha_iX_i|\} \leq \mathbb{E}\{|\sum_{j=1}^{m} \gamma_jX_j|\}. \]
To estimate this last expression it suffices to consider the extreme case $\gamma_j = 2^{-(j-1)/2}$, for $j = 1, \ldots, m$. In this case — applying again repeatedly the argument used to obtain (3):
\[
\mathbb{E}\{|\sum_{j=1}^{m} 2^{-(j-1)/2} X_j|\} \leq \mathbb{E}\{|\sum_{j=1}^{m-1} 2^{-(j-1)/2} X_j + 2^{-(m-1)/2} X_m|\} \\
\leq \mathbb{E}\{|\sum_{j=1}^{m-2} 2^{-(j-1)/2} X_j + 2^{-(m-2)/2} X_m|\} \\
\leq \mathbb{E}\{|X_1 + X_2|\} \leq \mathbb{E}\{|\sqrt{2}X_1|\} = \sqrt{2}\mathbb{E}\{|X_1|\}.
\]

**Step 3.** $\mathbb{E}\{X^2\} < \infty$.
We deduce from Step 2 that for a sequence $(\alpha_i)_{i=1}^{\infty}$ with $\sum_{i=1}^{\infty} \alpha_i^2 < \infty$ the series
\[
\sum_{i=1}^{\infty} \alpha_i X_i
\]
converges in mean and therefore almost surely. Using the notation
\[
[S] = \begin{cases} 
S & \text{if } |S| \leq 1, \\
\text{sign}(S) & \text{if } |S| \geq 1.
\end{cases}
\]
for a random variable $S$, we deduce from Kolmogorov’s three series theorem that
\[
\sum_{i=1}^{\infty} \mathbb{E}\{|\alpha_i X_i|\} < \infty.
\]
Suppose now that $\mathbb{E}\{X^2\} = \infty$; this implies that for every $C > 0$, we can find $\alpha > 0$ such that
\[
\mathbb{E}\{|\alpha X|^2\} \geq C\alpha^2.
\]
From this inequality it is straightforward to construct a sequence $(\alpha_i)_{i=1}^{\infty}$ such that
\[
\sum_{i=1}^{\infty} \mathbb{E}\{|\alpha_i X_i|\} = \infty, \text{ while } \sum_{i=1}^{\infty} \alpha_i^2 < \infty,
\]
a contradiction proving Step 3.

**Step 4.** Finally, we show how $\mathbb{E}\{X^2\} < \infty$ implies that $X$ is normal. We follow the argument of Bobkov and Houdré [2].
The finiteness of the second moment implies that we must have equality in the assumption of the theorem, i.e.,
\[
\mathbb{P}\{|X+Y| \geq \sqrt{2}t\} = \mathbb{P}\{|X| \geq t\}.
\]
Indeed, assuming that there is strict inequality in (1) for some $t > 0$, we would obtain that the second moment of $X + Y$ is strictly smaller than the second moment of $\sqrt{2}X$, which leads to a contradiction:
\[
2\mathbb{E}\{X^2\} > \mathbb{E}\{(X + Y)^2\} = \mathbb{E}\{X^2\} + \mathbb{E}\{Y^2\} = 2\mathbb{E}\{X^2\}.
\]
Hence, $2^{-n/2}(X_1 + \ldots + X_{2^n})$ has the same distribution as $X$ and we deduce from the Central Limit Theorem that $X$ is Gaussian.
References
