PERCOLATION OF ARBITRARY WORDS
ON THE CLOSE-PACKED GRAPH OF $\mathbb{Z}^2$

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Abstract Let $\mathbb{Z}_{cp}^2$ be the close-packed graph of $\mathbb{Z}^2$, that is, the graph obtained by adding to each face of $\mathbb{Z}^2$ its diagonal edges. We consider site percolation on $\mathbb{Z}_{cp}^2$, namely, for each $v$ we choose $X(v) = 1$ or 0 with probability $p$ or $1 - p$, respectively, independently for all vertices $v$ of $\mathbb{Z}_{cp}^2$. We say that a word $(\xi_1, \xi_2, \ldots) \in \{0, 1\}^\mathbb{N}$ is seen in the percolation configuration if there exists a selfavoiding path $(v_1, v_2, \ldots)$ on $\mathbb{Z}_{cp}^2$ with $X(v_i) = \xi_i, i \geq 1$. $p_c(\mathbb{Z}^2, \text{site})$ denotes the critical probability for site-percolation on $\mathbb{Z}^2$. We prove that for each fixed $p \in (1 - p_c(\mathbb{Z}^2, \text{site}), p_c(\mathbb{Z}^2, \text{site}))$, with probability 1 all words are seen. We also show that for some constants $C_i > 0$ there is a probability of at least $C_1$ that all words of length $C_0 n^2$ are seen along a path which starts at a neighbor of the origin and is contained in the square $[-n, n]^2$.

Keywords Percolation, close-packing.

AMS subject classification Primary. 60K35

1. Introduction.

Benjamini and Kesten (1995) introduced the problem whether ‘all words are seen in percolation on a graph $\mathcal{G}$’. The set-up is as follows. $\mathcal{G}$ is an infinite connected graph and the vertices of $\mathcal{G}$ are independently chosen to be occupied with probability $p$ and vacant with probability $1 - p$. The resulting probability measure on configurations of occupied and vacant vertices of $\mathcal{G}$ is denoted by $P_p$. We set

$$X(v) = \begin{cases} 1 & \text{if } v \text{ is occupied} \\ 0 & \text{if } v \text{ is vacant.} \end{cases}$$

Under $P_p$ the $X(v)$, $v$ a vertex of $\mathcal{G}$, are i.i.d. binomial variables with $P_p\{X(v) = 1\} = p$.

A path on $\mathcal{G}$ will be a sequence $\pi = (v_0, v_1, \ldots)$, with $v_0, v_1, \ldots$ vertices of $\mathcal{G}$, such that $v_i$ and $v_{i+1}$ are adjacent for $i \geq 0$. The path $\pi$ is called self-avoiding if all its vertices are distinct. A path may be finite or infinite. A word is an (finite or infinite) sequence $\xi = (\xi_1, \xi_2, \ldots)$ of zeroes and ones. A finite word $(\xi_1, \ldots, \xi_n)$ of $n$ zeroes and ones is said to have length $n$. The space of all infinite words is denoted by

$$\Xi = \{0, 1\}^\mathbb{N}.$$ 

We will say that the word $\xi = (\xi_1, \xi_2, \ldots)$, is seen along a path $\pi = (v_0, v_1, \ldots)$ on $\mathcal{G}$ if $\pi$ is self-avoiding and $X(v_i) = \xi_i$, $i \geq 1$ (note that the state of the initial vertex $v_0$ doesn’t figure in this definition).

Benjamini and Kesten (1995) investigated when all words in $\Xi$ are seen along some self-avoiding path on $\mathcal{G}$. They also considered the even stronger requirement that all words of $\Xi$ are seen along some self-avoiding path which starts at a fixed vertex $v_0$. They showed that even this stronger phenomenon occurs with positive probability when $p = 1/2$ and $\mathcal{G} = \mathbb{Z}^d$ for sufficiently high $d$. A weaker phenomenon is that almost all words from $\Xi$ are seen along some self-avoiding path. Here ‘almost all’ is with respect to some measure $\mu$ on $\Xi$. In all investigations so far one has taken $\mu$ to be a product measure

$$\mu = \prod_{i=1}^{\infty} \mu_i,$$ \hfill (1.1)

with $\mu(\{0\}) = 1 - \mu(\{1\}) = \beta$ for some $0 < \beta < 1$. Kesten, Sidoravicius and Zhang (1998) proved that if $p = 1/2$ and $\mathcal{G}$ is the triangular lattice, then almost all words are seen in this sense (for any $0 < \beta < 1$).

This paper deals only with the graph $\mathbb{Z}^2_{cp}$, which is obtained by ‘close packing’ the faces of $\mathbb{Z}^2$. Throughout this paper we think of the square lattice $\mathbb{Z}^2$ as being imbedded in $\mathbb{R}^2$ in the standard way. Let $F$ be a face of $\mathbb{Z}^2$. Close-packing $F$ means adding an edge to $\mathbb{Z}^2$ between any pair of vertices on the perimeter of $F$ which are not yet adjacent. The vertex set of $\mathbb{Z}^2_{cp}$ is therefore the same as the vertex set of $\mathbb{Z}^2$, and the edge set of $\mathbb{Z}^2_{cp}$ consists of the edge set of $\mathbb{Z}^2$ plus, for each face $F$ of $\mathbb{Z}^2$, two ‘diagonal’ edges between pairs of vertices on the perimeter of $F$. $\mathbb{Z}^2_{cp}$ and $\mathbb{Z}^2$ are a matching pair of graphs in the terminology of Kesten (1982), Section 2.2. (In the notation of that section $\mathbb{Z}^2_{cp}$ and $\mathbb{Z}^2$ are based on the ‘mosaic’ $\mathbb{Z}^2$ and the collection of all its faces.) Note that $\mathbb{Z}^2_{cp}$ is not a planar graph anymore. $p_c(\mathbb{Z}^2, \text{site})$ will denote the critical probability for site percolation on $\mathbb{Z}^2$. It is known that

$$p_c(\mathbb{Z}^2, \text{site}) \geq .556$$
(see van den Berg and Ermakov (1996)). Moreover, it follows from Theorem 3.1 and Corollary 3.1 in Kesten (1982) (see also pp. 54-56 there) or from Russo (1981) that the critical probability for site percolation on \( \mathbb{Z}^2_{cp} \) is \( 1 - p_c(\mathbb{Z}^2, \text{site}) \). Throughout we will fix \( p \) so that

\[
1 - p_c(\mathbb{Z}^2, \text{site}) < p < p_c(\mathbb{Z}^2, \text{site}).
\]  

(1.2)

In this note we will prove the following result.

**Theorem.** Let \( p \) satisfy (1.2). Then

\[
P_p \{ \text{all } \xi \in \Xi \text{ are seen along some path on } \mathbb{Z}^2_{cp} \text{ from the origin} \} > 0
\]  

(1.3)

and

\[
P_p \{ \text{all } \xi \in \Xi \text{ are seen along some path on } \mathbb{Z}^2_{cp} \} = 1.
\]  

(1.4)

Moreover, there exist constants \( C_i = C_i(p) > 0 \) such that

\[
P_p \{ \text{for all large } n \text{ all words of length } C_2n^2 \text{ are seen along some path on } \mathbb{Z}^2_{cp} \text{ from the origin and inside } [-n,n]^2 \} \geq C_1.
\]  

(1.5)

We note that the triangular lattice is a sublattice of \( \mathbb{Z}^2_{cp} \). Thus, at \( p = 1/2 \), the result of Kesten, Sidoravicius and Zhang (1998) already shows that almost all words (with respect to a measure \( \mu \) of the form (1.1)) are seen somewhere on \( \mathbb{Z}^2_{cp} \). (1.3) is of course a much stronger statement (which does not hold on the triangular lattice, even at \( p = 1/2 \)).

**Acknowledgement** The authors thank several institutes for their support and hospitality while this research was being carried out. In the case of H. K., the Inst. Hautes Etudes Scientifiques during May-July 1999; in the case of V. S., the University of Rouen during May-July 1999 and Cornell University during May 2000; in the case of Y. Z., IMPA/CNPq during May 1998. This research was also supported in part by NSF Grant # 9970943 to Cornell University and by NSF Grant # 9618128 to the University of Colorado, and Faperj Grant # E-26/150.940/99 and PRONEX.

2. “Double paths” and their properties.

It is convenient for our proofs to introduce a planar graph \( \mathcal{M} \) which is closely related to \( \mathbb{Z}^2_{cp} \). \( \mathcal{M} \) is obtained from \( \mathbb{Z}^2 \) by adding in each face \( F \) of \( \mathbb{Z}^2 \) a vertex which is connected by an edge to each of the four vertices on the perimeter of \( F \). We call the added vertices central vertices and to help us in picturing these we think of them as being located at the points \( (i + \frac{1}{2}, j + \frac{1}{2}), \; i, j \in \mathbb{Z}^2 \). Another way of picturing \( \mathcal{M} \) is to make the crossing of two diagonal edges, which are added to a face \( F \) when forming \( \mathbb{Z}^2_{cp} \) from \( \mathbb{Z}^2 \), into a vertex of \( \mathcal{M} \). A path and a self-avoiding path on \( \mathcal{M} \) are defined in the obvious way.

We shall frequently associate a self-avoiding path \( (w_1, w_2, \ldots) \) on \( \mathbb{Z}^2_{cp} \) to a self-avoiding path \( (v_0, v_1, \ldots) \) on \( \mathcal{M} \). This is done in the following unique way. We take

\[
w_1, w_2 \ldots \text{ as the successive noncentral vertices among } v_1, v_2, \ldots.
\]  

(2.1)
We regard the $w_i$ also as vertices on $\mathbb{Z}^2_{c_p}$. With this interpretation it is clear that $(w_1, w_2, \ldots)$ is a self-avoiding path on $\mathbb{Z}^2_{c_p}$. We call this path the path associated to $(v_0, v_1, \ldots)$. Note that $v_0$ never is a vertex of the associated path, even if $v_0$ is a noncentral vertex.

We will call a self-avoiding path $(v_0, v_1, \ldots, v_m)$ on $\mathcal{M}$ occupied (vacant) if and only if all noncentral vertices among the $v_i$ are occupied (respectively, vacant). The occupancy or vacancy of the central vertices will not be significant for our purposes, since we only want to discuss words seen on $\mathbb{Z}^2_{c_p}$. Accordingly, if $\pi = (v_0, v_1, \ldots)$ is a path on $\mathcal{M}$ and $\xi = (\xi_1, \xi_2, \ldots)$ is a word, then we say that $\xi$ is seen along $\pi$ (on $\mathcal{M}$) if and only if

$$X(w_i) = \xi_i, \ i \geq 1.$$  \hfill (2.2)

Intuitively speaking this says that $\xi$ is seen along the path associated to $\pi$. This is not quite accurate though, because when we consider whether $\xi$ is seen along a path on $\mathbb{Z}^2_{c_p}$ we ignore the initial point of the path. Because of this technicality, (2.2) really says that $X(w_1) = \xi_1$ and $(\xi_2, \xi_3, \ldots)$ is seen along the path associated to $\pi$.

Next we define a double path on $\mathcal{M}$. This is defined to be a pair of self-avoiding paths $\pi', \pi''$ on $\mathcal{M}$ which satisfy the following properties (2.3)-(2.7):

- $\pi'$ and $\pi''$ have no vertices in common;  \hfill (2.3)
- $\pi'$ is occupied and $\pi''$ is vacant;  \hfill (2.4)
- the initial points of $\pi'$ and $\pi''$, $u'$ and $u''$, are neighbors on $\mathcal{M}$;  \hfill (2.5)
- the final points of $\pi'$ and $\pi''$, $v'$ and $v''$, are neighbors on $\mathcal{M}$.  \hfill (2.6)

In addition we will require the minimality property (2.7) below. Let $\overline{\pi}'$ and $\overline{\pi}''$ be a pair of paths on $\mathcal{M}$ which satisfy (2.3)-(2.6) with $\pi'$ replaced by $\overline{\pi}'$ and $\pi''$ replaced by $\overline{\pi}''$. Denote by $R(\overline{\pi}', \overline{\pi}'')$ the interior of the Jordan curve formed by concatenating $\overline{\pi}'$, $\{v', v''\}$, (the reverse of) $\overline{\pi}''$ and $\{u'', u'\}$. Let $\overline{R}(\overline{\pi}', \overline{\pi}'')$ be the union of $R(\overline{\pi}', \overline{\pi}'')$ and its boundary (that is, the above Jordan curve). Then we further require that

$$\overline{R}(\overline{\pi}', \overline{\pi}'')$$

is minimal among all such $\overline{R}(\overline{\pi}', \overline{\pi}'')$, \hfill (2.7)

that is, there does not exist a pair $\overline{\pi}', \overline{\pi}''$ satisfying (2.3)-(2.6) and such that $\overline{R}(\overline{\pi}', \overline{\pi}'')$ is strictly contained in $R(\overline{\pi}', \overline{\pi}'')$.

In order to find double paths, the following observation will be useful. For any given pair $\overline{\pi}', \overline{\pi}''$ which satisfies (2.3) - (2.6), there exist at most finitely many pairs $\overline{\pi}', \overline{\pi}''$ of paths on $\mathcal{M}$ which also satisfy (2.3)-(2.6) as well as

$$\overline{R}(\overline{\pi}', \overline{\pi}'') \subset \overline{R}(\overline{\pi}', \overline{\pi}'').$$

(Throughout this paper $A \subset B$ will mean that $A$ is contained in $B$, but not necessarily strictly; thus $A = B$ is possible if $A \subset B$). Now, for any pair $\overline{\pi}', \overline{\pi}''$ which satisfy (2.3)-(2.6), there exists a pair $\pi', \pi''$ which satisfies (2.3)-(2.7) (that is, a double path) which in addition satisfies

$$\overline{R}(\pi', \pi'') \subset \overline{R}(\overline{\pi}', \overline{\pi}'').$$  \hfill (2.8)

The following lemma gives an important necessary condition for $\pi', \pi''$ to be a double path.
Lemma 1. If $\pi', \pi''$ is a double path, and if $v$ is a vertex on $\pi'$, then there exists a vertex $w$ adjacent (on $M$) to $v$, such that $w$ is connected by a vacant path on $M$ in $\overline{R}(\pi', \pi'') \setminus \pi' \to \pi''$.

Similarly, if $v$ is a vertex on $\pi''$, then there exists a vertex $w$ adjacent (on $M$) to $v$, such that $w$ is connected by an occupied path on $M$ in $\overline{R}(\pi', \pi'') \setminus \pi'' \to \pi'$.

Note that it is possible in the first part of the lemma that $w \in \pi''$. In this case, the vacant path from $w$ to $\pi''$ consists of $w$ only. A similar comment applies to the second part.

Proof of Lemma 1. Because of the symmetric roles of $\pi', \pi''$ we only need to prove the first part of the Lemma. We then apply Proposition 2.2 of Kesten (1982) with ‘occupied’ and ‘vacant’ interchanged (note that $J$ should be $\overline{J}$ on lines 1 and 2 from bottom on p. 30 of Kesten (1982)). We make the following choices: For the mosaic $M$ in Proposition 2.2 we take the present $M$. The graphs $G$ and $G^*$ are both taken equal to $M$ (this is indeed a matching pair based on the mosaic $M$, because all faces of $M$ are already close packed; see Def. 4 on p. 18 of Kesten (1982)). For the occupancy configuration on $M$ we extend the existing occupancy configuration on the noncentral vertices by declaring each central vertex on $\pi''$ to be vacant and all central vertices not on $\pi''$ to be occupied. $J$ is the Jordan curve formed by concatenating $\pi'$, $\{v', v''\}$, (the reverse of) $\pi''$ and $\{u'', u'\}$. Finally, $A_1 = \{v\}$, $A_2$ is the piece of $\pi'$ from $v$ (including $v$) to $v'$, followed by the edge $\{v', v''\}$, $A_3 = \pi''$, and finally $A_4$ is the edge $\{u'', u'\}$ followed by the piece of $\pi'$ from $u'$ to $v$ (including $v$). Then $\overline{J} = \overline{R}(\pi', \pi'')$ and by the minimality property (2.7) there does not exist an occupied path $r^* \in G = M$ inside $\overline{R}(\pi', \pi'') \setminus A_1 \cup A_3$ from a vertex of $A_2$ to a vertex of $A_4$. (As in Kesten (1982), $\hat{A}_i$ stands for $A_i$ minus its endpoints.) Indeed, if such a path $r^*$ would exist, then it would contain a crosscut $\tilde{r}$ of $R(\pi', \pi'')$ and we could change $\pi'$ to a new path $\tilde{\pi}'$ by replacing a piece of $\pi'$ by $\tilde{r}$, such that

$$\overline{R}(\pi', \pi'') \subset \overline{R}(\tilde{\pi}', \pi'').$$

(As in Newman (1951) we define a crosscut of $R$ to be a simple curve in $\overline{R}$ with only its endpoints on the boundary of $R$.) This would contradict (2.7). Proposition 2.2 of Kesten (1982) now gives the existence of a path $r \in G = M$ from $v$ to $\pi''$ in $\overline{R}(\pi', \pi'') \setminus A_1 \cup A_3$ such that all vertices of $r$ in $\overline{J} \setminus A_1 \cup A_3$ are vacant. In fact, $r$ minus its initial point $v$ must be disjoint from $\pi'$, because all vertices on $\pi'$ are occupied. We now take for $w$ the first vertex on $r$ after $v$. The path required in the lemma is then the piece of $r$ from $w$ to $\pi''$. \(\square\)

Now let $(\pi', \pi'')$ be a double path starting at $(u', u'')$ and ending at $(v', v'')$, and define

$$\Theta = \Theta(\pi', \pi'') = \left[\frac{1}{4} \min(||v' - u'||, ||v'' - u''||)\right], \quad (2.9)$$

where $|| \cdot ||$ denotes the Euclidean norm. (This definition differs slightly from the corresponding one in Kesten, Sidoravicius and Zhang (1998).) Finally, let $\xi = (\xi_1, \ldots) \in \Xi$ be any infinite word. The next proposition (which is purely deterministic) shows how one can ‘see’ an initial segment of the word $\xi$ inside $\overline{R}(\pi', \pi'')$. This proposition is our principal technical step. It is entirely analogous to Lemma 8 in Kesten, Sidoravicius and Zhang (1998).
Proposition 2. Let \((\pi', \pi'')\) be a double path starting at \((u', u'')\) and ending at \((v', v'')\) and let \(\xi \in \Xi\) be arbitrary. Then there exist paths \(\sigma' = (\sigma'_0 = u', \sigma'_1, \ldots), \sigma'' = (\sigma''_0 = u'', \sigma''_1, \ldots)\) on \(\mathcal{M}\) with the following properties:

\[
\sigma', \sigma'' \subset \overline{R}(\pi', \pi''); \quad (2.10)
\]

\[
\sigma' \text{ and } \sigma'' \text{ start at } u' \text{ and } u'', \text{ respectively}; \quad (2.11)
\]

\[
\text{the endpoints of } \sigma' \text{ and } \sigma'' \text{ belong to } \{v', v''\}; \quad (2.12)
\]

one sees an initial segment of \(\xi\) on \(\mathcal{M}\) from \(u'(u'')\) along \(\sigma'(\sigma'')\). This initial segment of \(\xi\) contains at least \((\xi_1, \ldots, \xi_{\Theta - 1})\). \(\quad (2.13)\)

Proof. We only prove that we can find the path \(\sigma'\) from \(u'\). The argument for \(\sigma''\) is the same except for an interchange of the roles of ‘occupied’ and ‘vacant’. For brevity we shall suppress the primes on \(\sigma\). We have to show that one can choose a path \(\sigma = (\sigma_0 = u', \sigma_1, \ldots, \sigma_\nu)\) on \(\mathcal{M}\) contained in \(\overline{R}(\pi', \pi'')\), such that \(\sigma_1\) is adjacent to \(u', \sigma_\nu \in \{v', v''\}\), and such that along \(\sigma\) one sees on \(\mathcal{M}\) an initial segment containing at least the first \(\Theta - 1\) components of \(\xi\). We prove this in the following recursive way. The steps differ slightly, depending on whether \(u'\) is a central vertex or not. If \(u'\) is a central vertex, then we find a vertex \(\sigma_1\) of \(\mathbb{Z}_c^{2p}\) which is adjacent on \(\mathcal{M}\) to \(u'\) and which lies in \(\overline{R}(\pi', \pi'')\) and is such that \(\sigma_1\) is occupied (vacant) if \(\xi_1 = 1\) \((\xi_1 = 0)\). We further find a new double path \((\pi'_1, \pi''_1)\) with \(\sigma_1\) the initial point of one of them and with endpoints \((v', v'')\), and such that

\[
\overline{R}(\pi'_1, \pi''_1) \subset \overline{R}(\pi', \pi'') \setminus \{u'\}. \quad (2.14)
\]

If \(u'\) is noncentral, then we either use the same construction as just outlined or take \(\sigma_1, \sigma_2\) such that \(\sigma_1\) is a central vertex adjacent on \(\mathcal{M}\) to \(u'\) and such that \(\sigma_2\) is adjacent on \(\mathcal{M}\) to \(\sigma_1\). Moreover \(\sigma_2\) has to be a vertex of \(\mathbb{Z}_c^{2p}\) which lies in \(\overline{R}(\pi', \pi'')\) and such that \(\sigma_2\) is occupied (vacant) if \(\xi_1 = 1\) \((\xi_1 = 0)\). This time we find a new double path \((\pi'_1, \pi''_1)\) with \(\sigma_2\) the initial point of one of them and with endpoints \((v', v'')\), and such that

\[
\overline{R}(\pi'_1, \pi''_1) \subset \overline{R}(\pi', \pi'') \setminus \{u', \sigma_1\} \quad (2.15)
\]

holds (instead of (2.14)).

We then repeat this step with \((\pi'_1, \pi''_1)\) replacing \((\pi', \pi'')\). This construction will continue until for the first time we come to a double path \((\pi'_\nu, \pi''_\nu)\) with

\[
\Theta(\pi'_\nu, \pi''_\nu) \leq 1.
\]

Our construction is such that the initial point of one of \(\pi'_{k+1}, \pi''_{k+1}\) is within distance 2 of one of the initial points of \(\pi'_k, \pi''_k\), and the endpoints of all the double paths are \(v'\) and \(v''\). Therefore,

\[
\Theta(\pi'_{k+1}, \pi''_{k+1}) \geq \Theta(\pi'_k, \pi''_k) - 1
\]
and it takes at least $\Theta(\pi', \pi'') - 1$ steps before we stop. Since at each recursive step one $\xi_j$ is used, it is clear that we will see at least $(\xi_1, \ldots, \xi_{\Theta-1})$ when our process stops.

Several cases have to be distinguished in our construction, depending on the value of $\xi_1$ and on whether $u'$ is a central vertex or not.

**Case (Ia)** $\xi_1 = 1$ and $u'$ is a central vertex. This case is treated in exactly the same way as Case (i) in Lemma 8 of Kesten, Sidoravicius and Zhang (1998). We briefly state the essentials. We take $\sigma_1$ to be the first vertex of $\pi' \setminus \{u'\}$. Since $u'$ is a central vertex, $\sigma_1$ is necessarily a vertex of $\mathbb{Z}^2_{cp}$. Since $\sigma_1 \in \pi'$, it is occupied, in agreement with the requirement $X(\sigma_1) = \xi_1 = 1$.

We next take $\widehat{\pi}'$ to be the piece of $\pi'$ from $\sigma_1$ to $v'$ (this is just $\pi'$ minus its first edge). We also want a vacant path $\widehat{\pi}''$. To choose this, we observe that by Lemma 1 there must exist a vacant path $\pi_2$ on $\mathcal{M}$ from a neighbor $u_1$ on $\mathcal{M}$ of $\sigma_1$ to $\pi''$, and such that

$$\pi_2 \in \overline{R}(\pi', \pi'') \setminus \pi'. \quad (2.16)$$

Now form the vacant path $\widehat{\pi}''$ on $\mathcal{M}$ from $u_1$ to $v''$ which consists of $\pi_2$ followed by the piece of $\pi''$ from the endpoint of $\pi_2$ to $v''$. Then $(\widehat{\pi}', \widehat{\pi}'')$ is a pair of paths, occupied and vacant, respectively, from $(\sigma_1, u_1)$ to $(v', v'')$. $\widehat{\pi}'$ and $\widehat{\pi}''$ have no vertex in common, by virtue of (2.16) and (2.3). Finally, by construction, $\widehat{\pi}', \widehat{\pi}''$ and $\{\sigma_1, u_1\}$ are contained in $\overline{R}(\pi', \pi'') \setminus \{u'\}$, so that

$$\overline{R}(\widehat{\pi}', \widehat{\pi}'') \subset \overline{R}(\pi', \pi'') \setminus \{u'\}. \quad (2.17)$$

It is not clear that $(\widehat{\pi}', \widehat{\pi}'')$ itself has the minimality property corresponding to (2.7). However, using the observation before Lemma 1, we can take for $(\pi'_1, \pi''_1)$ an occupied and vacant pair of paths from $(\sigma_1, u_1)$ to $(v', v'')$ which makes $\overline{R}(\pi'_1, \pi''_1)$ minimal. This will automatically satisfy

$$\overline{R}(\pi'_1, \pi''_1) \subset \overline{R}(\pi', \pi'') \subset \overline{R}(\pi', \pi'') \setminus \{u'\}. \quad (2.17)$$

Thus (2.14) will be satisfied and we are done with our recursive step in Case (Ia).

**Case (Ib)** $\xi_1 = 1$ and $u'$ is not a central vertex. Now $u'$ is a vertex of $\mathbb{Z}^2_{cp}$. It may be that the first vertex of $\pi' \setminus \{u'\}$ is also a vertex of $\mathbb{Z}^2_{cp}$. Then we take $\sigma_1$ equal to this first vertex of $\pi' \setminus \{u'\}$ and proceed as in case (Ia).

The only other possibility is that $\pi'$ begins with $u', u_1, w_2$ with $w_1$ some central vertex and $w_2$ some vertex of $\mathbb{Z}^2_{cp}$, such that $u'$ and $w_2$ are adjacent on $\mathbb{Z}^2_{cp}$, but not necessarily on $\mathcal{M}$. In this case we take $\sigma_1 = w_1$, $\sigma_2 = w_2$. We then again proceed as in case (Ia), but now with $\sigma_1$ replaced by $\sigma_2$. That is, we take for $\widehat{\pi}'$ the piece of $\pi'$ from $\sigma_2$ to $v'$. We further form $\widehat{\pi}''$ by concatenating a vacant path in $\overline{R}(\pi', \pi'') \setminus \pi'$ from a neighbor of $\sigma_2$ to $\pi''$ with a piece of $\pi''$ ending at $v'$. We then find the new double path $(\pi'_1, \pi''_1)$ which satisfies (2.15) by applying the observation before Lemma 1, as in Case (Ia).

**Case (IIa)** $\xi_1 = 0$ and $u'$ is a central vertex. This case closely follows case (ii) of Lemma 8 in Kesten, Sidoravicius and Zhang (1998). By a translation we may assume that $u' = (\frac{1}{2}, \frac{1}{2})$. We define $H$ to be the unit square centered at $u'$, that is,

$$H = \{(x, y) : |x - \frac{1}{2}| < \frac{1}{2}, |y - \frac{1}{2}| < \frac{1}{2}\}.$$
We denote the first vertex of $\pi' \setminus \{u'\}$ by $u_1$ and consider the four neighbors of $u'$ on $\mathcal{M}$. These are all vertices of $\mathbb{Z}^2_{cp}$ at the four ‘corners’ of $H$ (see Figure 1).

One of them is $u_1$ (which is occupied) and another one is $u''$ (which is vacant). Without loss of generality we take $u_1 = (1,0)$. Now the interior of one of the triangles with vertices $u', u_1$ and $(0,0)$ or with vertices $u', u_1$ and $(1,1)$ must be contained in $R(\pi', \pi'')$. For the sake of argument assume that

the interior of the triangle with vertices at $u', u_1$ and $(0,0)$

is contained in $R(\pi', \pi'')$.

Write $u_2$ for the third vertex of this triangle, that is, $u_2 = (0,0)$. We claim that $u_2$ must be vacant. Assume, to arrive at a contradiction, that $u_2$ is occupied (and hence $u_2 \notin \pi''$). If $u_2$ is not a vertex on $\pi'$, then we can replace the first edge of $\pi'$ (i.e., the edge $\{u', u_1\}$) by the two edges $\{u', u_2\}, \{u_2, u_1\}$. This will remove the triangle with vertices $u', u_1, u_2$ from $R(\pi', \pi'')$, in contradiction to the minimality property (2.7). If, on the other hand, $u_2$ is a vertex of $\pi'$, then we can replace the piece of $\pi'$ from $u'$ to $u_2$ by the single edge $\{u', u_2\}$. Since this last edge is a crosscut of $R(\pi', \pi'')$, this replacement will again strictly decrease $R(\pi', \pi'')$. This is impossible, again by (2.7). This proves our claim that $u_2$ is vacant.

We next claim that there exists a vacant connection $r$ on $\mathcal{M}$ from $u_2$ to $\pi''$. It may be that $u_2$ already lies on $\pi''$, in which case the sought vacant connection $r$ consists of $\{u_2\}$ only. To see that $r$ exists in general, we move along the arc of the perimeter of $H$ from $u_2$ to $u''$ which does not contain $u_1$. Let $u_3$ be the first vertex of $\pi' \cup \pi''$ we meet while moving along this arc and denote by $A_2$ the piece from $u_2$ to $u_3$ of this arc (including its endpoints $u_2$ and $u_3$). As in case (ii) of Lemma 8 in Kesten, Sidoravicius and Zhang (1998) it cannot be the case that $u_3 \in \pi'$. Indeed, if $u_3$ were a vertex of $\pi'$, then the edge $\{u', u_3\}$ would be a crosscut of $R(\pi', \pi'')$, because its endpoints would lie on $\pi'$, and it could be connected in $H$ to $A_2 \subset R(\pi', \pi'')$ without intersecting $\pi' \cup \pi''$. But if $\{u', u_3\}$ is a crosscut of $R(\pi', \pi'')$, then we can replace the piece of $\pi'$ from $u'$ to $u_3$ by the single edge $\{u', u_3\}$ and strictly decrease $R(\pi', \pi'')$. Since this contradicts (2.7), we must have

- For the interior of $R(\pi', \pi'')$.
- For the boundary of $H$.
- For the triangle with vertices $u_0, u_1, (0,0)$.
- For the triangle with vertices $u_0, u_1, (1,1)$.
- For the crosscut of $R(\pi', \pi'')$.
- For the replacement of the piece from $u'$ to $u_2$ by the single edge $\{u', u_2\}$.
- For the strict decrease of $R(\pi', \pi'')$.
- For the contradiction of (2.7).
$u_3 \in \pi''$. If $A_2$ is vacant, then we can take $r = A_2$. In the other cases we prove the existence of $r$ by an application of Proposition 2.2 of Kesten (1982). We make the following choices. For the occupancy configuration of $\mathcal{M}$ we extend the occupancy configuration on $\mathbb{Z}^2$ by taking all central vertices on $\pi''$ vacant and all vertices off $\pi''$ occupied. Further we take $A_1 = \{u_2\}$, $A_2$ as above, $A_3 = \text{the piece of } \pi''$ from $u_3$ to $v''$, and $A_4 = \text{the piece of } \{v'', w\}$, followed by the piece of (the reverse of) $\pi'$ from $v'$ to $u_1$, followed by the edge $\{u_1, u_2\}$. These arcs make up a Jordan curve $J$ which is contained in $\overline{R(\pi', \pi'')}$. Therefore the interior of $J$ is contained in $R(\pi', \pi'')$. Moreover, there cannot exist in $\overline{J}$ an occupied path $r^*$ on $\mathcal{M}$ from some vertex $w$ in $\partial A_2$ to some vertex in $\partial A_4$ (a vertex in $\partial A_4$ is a vertex in $\pi'$). Indeed, since $w \in A_2$, $w$ would be a neighbor on $\mathcal{M}$ of $u'$. Therefore, the edge $\{u', w\}$ followed by such a path $r^*$ would form an occupied crosscut of $R(\pi', \pi'')$, and no such crosscut can exist by the minimality property (2.7). We can therefore apply Proposition 2.2 of Kesten (1982) (with ‘occupied’ and ‘vacant’ interchanged and corrected by replacing $J$ by $\overline{J}$ in lines 1 and 2 from bottom on p. 30). This guarantees the existence of a vacant path $r$ on $\mathcal{M}$ in $\overline{J} \subset \overline{R(\pi', \pi'')}$. $r$ is necessarily disjoint from $\pi'$, since all vertices on $\pi'$ are occupied and all vertices on $r$ are vacant. This proves that $r$ has the desired properties.

We can now complete our choices for this recursive step. We take $\sigma_1 = u_2$ (which satisfies the requirement $X(\sigma_1) = 0 = \xi_1$). We further take $\pi'_1 = \text{piece of } \pi'$ from $u_1$ to $v'$, and $\pi'' = \text{the path } r \text{ followed by the piece of } \pi'' \text{ from the endpoint of } r \text{ to } v''$. Finally, we again choose $\pi'_1, \pi''_1$ such that (2.17) holds and such that $\overline{R(\pi'_1, \pi''_1)}$ is minimal in the family of possible $\overline{R(\pi', \pi'')}$ satisfying (2.17).

Case (IIb) $\xi_1 = 0$ and $u'$ is not a central vertex, but $u_1 := \text{first vertex of } \pi' \setminus \{u'\}$ is central. The argument is similar to that of the last case. By translation and rotation we may assume that $u' = (0, 0)$ and $u_1 = (\frac{1}{2}, \frac{1}{2})$. One ‘side’ of the edge $\{u', u_1\}$ must lie in $R(\pi', \pi'')$. By symmetry we may therefore assume that the interior of the triangle with vertices $u', u_1$ and $(1, 0)$ is contained in $R(\pi', \pi'')$.

We now follow the argument of the preceding case. This time we take $H$ to be the ‘diamond’ $\{(x, y) : |x| + |y| = 1\}$ (see Figure 2).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diamond.png}
\caption{The solidly drawn diamond is the boundary of $H$. The interior of the hatched triangle is contained in $R(\pi', \pi'')$.}
\end{figure}

$u_2$ will denote the vertex $(1, 0)$. The boundary of $H$ is a Jordan curve which necessarily contains the initial point $u''$ of $\pi''$. When one moves from $u_2$ to $u''$
along the arc of the boundary of $H$ which does not contain $u_1$, then one will meet a first vertex, $u_3$ say, of $\pi' \cup \pi''$. We denote by $A_2$ the arc of the boundary of $H$ from $u_2$ to $u_3$. As in the preceding case we now show that $u_2$ must be vacant and $u_3 \in \pi''$. Moreover, $u_2$ must have a vacant connection $r$ on $\mathcal{M}$ in $\overline{R}(\pi', \pi'')$ to $\pi''$.

Finally we complete this recursive step when $u_1 = (\frac{3}{2}, \frac{1}{2})$, by taking $\sigma_1 = u_2$, $\widehat{\pi}'$ = the piece of $\pi'$ from $u_1$ to $v'$ and $\widehat{\pi}''$ = the path $r$ followed by the piece of $\pi''$ from the endpoint of $r$ to $v''$.

**Case (IIc)** $\xi_1 = 0$ and neither $u'$ nor $u_1$ are central. Now we may assume that $u' = (0, 0)$ and $u_1 = (1, 0)$. In this case there is again a triangle adjacent to the edge $\{u', u_1\}$ whose interior is contained in $R(\pi', \pi'')$. Without loss of generality, let this be the triangle with vertices $u', u_1$ and $(\frac{1}{2}, -\frac{1}{2})$. We then take $u_2 = (\frac{3}{2}, -\frac{1}{2})$. This vertex does not lie on $\pi'$ for the same reasons as in case (IIa). It is a central vertex. Since it does not lie on $\pi'$ we may declare this site to be vacant. We further declare all central vertices on $\pi''$ vacant and all central vertices off $\pi''$ other than $u_2$ occupied. We can now use the same proof as in case (IIb) or case (IIa) to show that there exists a vacant connection $r$ on $\mathcal{M}$ in $\overline{R}(\pi', \pi'') \setminus \pi'$ from $u_2$ to $\pi''$. However, $\sigma_1$ and the paths $\widehat{\pi}', \widehat{\pi}''$ have to be chosen slightly differently than in the previous cases. Let $\widehat{\pi}$ be the (vacant) path consisting of $r$ and the piece of $\pi''$ from the endpoint of $r$ to $v'$. Denote the first vertex of $\widehat{\pi}$ after its initial point by $\rho$ and take $\widehat{\pi}''$ to be the piece of $\widehat{\pi}$ from $\rho$ to $v''$. Thus, basically, $\widehat{\pi}''$ equals $\widehat{\pi}$ minus its first edge.

Since $r$ starts at $(\frac{3}{2}, -\frac{1}{2})$ and $u' = (0, 0)$ and $u_1 = (1, 0)$ lie on $\pi'$, $\rho$ can only take the values $(0, -1)$ and $(1, -1)$. If $\rho = (1, -1)$, then $\rho$ is adjacent to $u_1$ on $\mathcal{M}$. We now take $\sigma_1 = u_2 = (\frac{3}{2}, -\frac{1}{2}), \sigma_2 = \rho = (1, -1), \widehat{\pi}'$ = the piece of $\pi'$ from $u_1$ to $v'$, and $\widehat{\pi}'' = \widehat{\pi}$. Then (2.15) holds. This finishes the recursive step when $\rho = (1, -1)$.

If $\rho = (0, -1)$, then $\rho$ is not adjacent to $u_1$ on $\mathcal{M}$. In this situation we therefore take for $\widehat{\pi}'$ the path consisting of the edge from $u_2$ to $u_1$ followed by the piece of $\pi'$ from $u_1$ to $v'$. We further take $\sigma_1 = \rho = (0, -1)$ and $\widehat{\pi}'' = \widehat{\pi}$, as before. Now $\widehat{\pi}'$ and $\widehat{\pi}''$ again start at adjacent points on $\mathcal{M}$ (namely at $u_2$ and $\rho = \sigma_1$) and end at $v'$ and $v''$. Also (2.14) holds. Moreover $\sigma_1$ is adjacent to $u'$ on $\mathcal{M}$ and $X(\sigma_1) = 0$, as required. This finishes the recursive step for this last case. \[\square\]

3. **Proof of the first part of Theorem 1.**

With proposition 2 we can prove (1.3) and (1.4) quickly from known facts about supercritical percolation. Even though (1.3), and consequently also (1.4), is contained in (1.5), we give first a direct proof of (1.3) and (1.4), because this is much easier than (1.5), and most readers will be satisfied with (1.3) and (1.4). The proof of (1.5) will be given in a separate section.

Throughout this section $C_i$ will denote a strictly positive, finite constant (independent of $n$). Define the following events:

$$E_1 = \{\text{there exists an infinite occupied path } \widehat{\pi}' \text{ on } \mathcal{M} \text{ from the origin}$$

in (the fourth quadrant) $\cup 0 = (0, \infty) \times (-\infty, 0) \cup \{0\} \},$

$$E_2 = \{\text{there exists an infinite vacant path } \widehat{\pi}'' \text{ on } \mathcal{M} \text{ from a neighbor}$$

of the origin and inside the first quadrant $(0, \infty)^2 \},$
and

\[ E_3 = \{ \text{for infinitely many } k \text{ there exists a vacant circuit on } M \text{ surrounding the origin in the annulus } [-2^{k+1}, 2^{k+1}]^2 \setminus [-2^k, 2^k]^2 \}. \]

Then (1.2) implies that \( P_p \{ E_1 \} > 0, P_p \{ E_2 \} > 0 \) and \( P_p \{ E_3 \} = 1 \) (see Grimmett (1999), Theorem 11.70, Smythe and Wierman (1978), Section 3.4). Since \( E_1 \) and \( E_2 \) are defined in terms of disjoint sets of vertices, they are independent and

\[ P_p \{ E_1 \cap E_2 \cap E_3 \} = P_p \{ E_1 \cap E_2 \} = P_p \{ E_1 \} P_p \{ E_2 \} > 0. \quad (3.1) \]

Now assume that \( E_1 \cap E_2 \cap E_3 \) occurs, and that \( \pi', \pi'' \) are an occupied and a vacant path as described in \( E_1 \) and \( E_2 \), respectively. Let \( u'' \) be the initial point of \( \pi'' \). Also let \( C_k \) be a vacant circuit surrounding the origin in \([-2^{k+1}, 2^{k+1}]^2 \setminus [-2^k, 2^k]^2 \). Then both paths \( \pi' \) and \( \pi'' \) must intersect \( C_k \). One can then construct from a piece of \( \pi'' \) and a piece of \( C_k \) a vacant path \( \pi'' \) on \( M \) from \( u'' \) to a vertex \( v'' \), adjacent on \( M \) to a vertex \( v' \in \pi' \). We can and will even choose \( \pi' \) disjoint from \( \pi'' \). As in the argument following (2.6) there then exists a double path \((\pi', \pi'')\) from \((u' = 0, u'')\) to \((v', v'')\) so that (2.8) holds. By Proposition 2 there then exists for each infinite word \( \xi \) a path \( \sigma \) from \( 0 = u' \) to \( v' \) or \( v'' \) such that an initial piece of \( \xi \) is seen along \( \sigma \). Since \( v' \in C_k \), any path from \( 0 \) to \( v' \) or \( v'' \) must contain at least \( 2^k + 1 \) noncentral vertices. Thus the length of the piece of \( \xi \) which is seen along \( \sigma \) is at least \( 2^k \).

This argument works for all \( \xi \) and all \( k \). Therefore, the left hand side of (1.3) is at least \( P_p \{ E_1 \cap E_2 \cap E_3 \} > 0 \). This proves (1.3). In turn, (1.4) then follows from (1.3) and the ergodic theorem (compare Harris (1960), Lemmas 3.1 and 5.1).

4. Proof of (1.5).

Before we turn to the details of the proof of (1.5) we give a brief outline. The proof is based on the construction of a “snake” inside of which the finite words will be seen. A snake will be the region \( R(\pi', \pi'') \) between an occupied path \( \pi' \) and a vacant path \( \pi'' \) which “wiggle a lot” (see Figure 3). More specifically, if \( u', v' (u'', v'') \) are the initial and endpoint of \( \pi' \) (of \( \pi'' \), respectively), then any path on \( M \) from \( u' \) to \( \{ v', v'' \} \) and contained in \( R(\pi', \pi'') \) will be forced to go back and forth between the strips \([0, n/4] \times \mathbb{R} \) and \([3n/4, n] \times \mathbb{R} \) at least \( C_3 n \) times. Thus, any such path will have length at least \( C_3 n \cdot n/2 \). If \((\pi', \pi'')\) is a double path from \((u', u'')\) to \((v', v'')\) with \( R(\pi', \pi'') \subset R(\pi', \pi'') \), then an initial piece of any word can be seen along a path from \( u' \) to \( (v', v'') \) in \( R(\pi', \pi'') \) (by Proposition 2). Because such a path must have length at least \( C_3 n^2 / 2 \), we will actually see any word of length \( C_4 n^2 \) along some path from \( u' \) to \( (v', v'') \) inside \( R(\pi', \pi'') \).
A somewhat more topological description of the situation is as follows. Let $J_1$ be the Jordan curve formed by concatenating $\pi', \{v', v''\}$, the reverse of $\pi''$ and $\{u'', u'\}$. $R(\pi', \pi'')$ is the interior of $J_1$. If $J$ is a Jordan curve, we will denote its interior by $J$. (This should not be confused with $\hat{A}$ when $A$ is an arc as in Section 2.) The way we will force a path from $u'$ to $\{v', v''\}$ to go back and forth between the strips $[0, n/4] \times \mathbb{R}$ and $[3n/4, n] \times \mathbb{R}$ is by constructing disjoint crosscuts $\lambda_{2i-1}, 1 \leq i \leq C3n$, from $\pi'$ to $\pi''$ in $R(\pi', \pi'')$ with $\lambda_{2i-1} \subset [0, n/4] \times \mathbb{R}$ when $i$ is even, and $\lambda_{2i-1} \subset [3n/4, n] \times \mathbb{R}$ when $i$ is odd. Each such crosscut $\lambda_{2i-1}$ divides $J_1$ into two components, one of which has $\{u', u''\}$ in its boundary, and the other of which has $\{v', v''\}$ in its boundary. This will force a path from $u'$ to $\{v', v''\}$ to cross each $\lambda_{2i-1}$. In fact, these $\lambda$'s will be constructed in such a way that a path from $u'$ to $\{v', v''\}$ has to cross $\lambda_1, \lambda_3, \ldots$ in succession. Because the crosscuts $\lambda_{2i-1}$ are alternately located in the strips $[0, n/4] \times \mathbb{R}$ and $[3n/4, n] \times \mathbb{R}$ this will also force any path from $u'$ to $\{v', v''\}$ to go back and forth between these strips, as desired.

In our construction, we will take $u'$ = the origin and the crosscuts $\lambda_{2i-1}$ will be constructed from appropriate left-right and top-bottom crossings of $[-n, n]^2$. The next lemma gives the deterministic (topological) part of the proof. As usual, the details of these arguments are messier than the simple intuitive picture suggests.

The last lemma will then estimate the probability that the various required crossings of $[-n, n]^2$ exist. Let $R = [a, b] \times [c, d]$ be a rectangle and $\hat{R} = (a, b) \times (c, d)$ be its interior. A left-right crossing of $R$ is a self-avoiding path on $M$ inside $[a, b] \times (c, d)$ with one endpoint on $\{a\} \times (c, d)$ and the other endpoint on $\{b\} \times (c, d)$. If $p$ is a left-right crossing of $R$, then $\hat{R} \setminus p$ consists of two components which contain $[a, b] \times \{d\}$ and $[a, b] \times \{c\}$ in their boundary, respectively. We shall denote these components by $p^+ = p^+(R)$ and $p^- = p^-(R)$, respectively. In a similar way one defines a top-bottom crossing of $R$ and its left and right component. If $\sigma$ is the top-bottom crossing, its left and right component will be denoted by $\sigma^\ell = \sigma^\ell(R)$ and $\sigma^r = \sigma^r(R)$, respectively. All paths in the next lemma are paths on $M$.

**Lemma 2.** Let $S(n) = [-n, n]^2$ and let $n$ be divisible by 4. Assume that the following conditions (4.1)-(4.5) are satisfied:

\[ \exists \text{ left-right crossings } \rho_1, \ldots, \rho_{4k-1} \text{ of the rectangle } [-n, n] \times [0, n] \]  

\[ \text{ (4.1)} \]
such that

\[ \text{all } \rho_i \text{ are disjoint,} \quad (4.2) \]

\[ \rho_i \text{ is vacant if } i \equiv 0, 3 \pmod{4}, \]
\[ \rho_i \text{ is occupied if } i \equiv 1, 2 \pmod{4}, \quad (4.3) \]
\[ \rho_{i+1}^+(S(n)) \subset \rho_i^+(S(n)), \quad 1 \leq i \leq 4k - 2, \quad (4.4) \]

and

\[ \rho_i \text{ has only one point on the vertical line } x = n/4 \text{ if } i \equiv 1, 2 \pmod{4} \]
\[ \text{and has only one point on the vertical line } x = 3n/4 \text{ if } i \equiv 0, 3 \pmod{4}. \quad (4.5) \]

Assume further that

\[ \exists \text{ a top-bottom crossings } \tau_1 \text{ of } [0, n/4] \times [-n, n] \]
\[ \text{and top-bottom crossings } \tau_2, \tau_3 \text{ of } [3n/4, n] \times [-n, n] \quad (4.6) \]

such that

\[ \tau_1, \tau_2, \tau_3 \text{ are disjoint,} \quad (4.7) \]
\[ \tau_1 \text{ and } \tau_3 \text{ are occupied and } \tau_2 \text{ is vacant,} \quad (4.8) \]

and

\[ \tau_3^i([3n/4, n] \times [-n, n]) \subset \tau_2^i([3n/4, n] \times [-n, n]). \quad (4.9) \]

Finally, assume that there exists an occupied path \( \gamma \) on \( M \) from \( u' = 0 \) to the vertical line \( x = n \) inside \( [0, n]\times[-n,0] \) and a vacant path \( \delta \) on \( M \) from a neighbor \( u'' \) of \( u' \) to the horizontal line \( y = n \) inside \( (-n, 0) \times [0, n] \). Then there exist a constant \( C_5 > 0 \), an occupied path \( \bar{\tau}' \) on \( M \) from \( u' \) to some \( v' \) and a vacant path \( \bar{\tau}'' \) on \( M \) from \( u'' \) to a neighbor \( v'' \) of \( v' \) such that

\[ \bar{\tau}' \text{ and } \bar{\tau}'' \text{ are disjoint,} \quad (4.10) \]
\[ \mathcal{R}(\bar{\tau}', \bar{\tau}'') \subset S(n), \quad (4.11) \]

and

\[ \text{every path from } u' \text{ to } \{v', v''\} \text{ inside } \mathcal{R}(\bar{\tau}', \bar{\tau}'') \text{ has length } \geq C_5 n(k - 1). \quad (4.12) \]

**Proof.** This proof will be broken down into a number of steps.

**Step 1.** This step is a simple observation which we shall use repeatedly. Let \( J \) be a Jordan curve with interior \( \mathring{J} \), and let \( \tau \) be a crosscut of \( \mathring{J} \) with endpoints \( a \) and \( b \) on \( J \). Then \( J \) is made up of two closed arcs from \( a \) to \( b \), which have only the points \( a \) and \( b \) in common. If these two arcs are denoted by \( A_1, A_2 \), then \( \mathring{J} \setminus \tau \) consists of two components, \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) say, and the boundary of \( \mathcal{K}_i \) consists of \( A_i \cup \tau \) (see
Newman (1951), Theorem V.11.8). Now let $\sigma$ be a further simple curve with one endpoint, $c$, on $\tau$ and the other endpoint, $d$, on $J$, such that

$$\sigma \setminus \{c, d\} \subset J \setminus \tau.$$  \hfill (4.13)

Then

$$d \in A_i \setminus \{a, b\}$$ implies that $\sigma$ is a crosscut of $K_i$. \hfill (4.14)

This is rather obvious, because $\sigma \setminus \{c, d\} \subset K_1 \cup K_2$ by virtue of (4.13). Also by (4.13), $\sigma \setminus \{c, d\}$ does not intersect the boundary of $K_1$, nor the boundary of $K_2$. Thus $\sigma \setminus \{c, d\}$ is entirely contained in $K_1$ or entirely contained in $K_2$. But if $d \in A_i \setminus \{a, b\}$, then $d$ lies only in the boundary of $K_i$ and therefore near $d$, $\sigma$ cannot contain points of $K_{3-i}$. Thus $\sigma \setminus \{c, d\} \subset K_i$, which proves (4.14).

**Step 2.** In this step we construct a first Jordan curve which will surround our snake. This Jordan curve will consist of pieces of $\gamma$, $\delta$, $\tau_3$ and $\rho_{4k-1}$. We begin with a Jordan curve $J_1$ which is constructed by concatenating the following curves: $\gamma$ from $u' = 0$ to its endpoint $a_1$ on $\{n\} \times [-n, n]$, the segment of $\{n\} \times [-n, n]$ from $a_1$ to the upper right hand corner of $S(n)$, $(n, n)$, the segment, $s$ say, of $[-n, n] \times \{n\}$ from $(n, n)$ to the endpoint $b_1$ of $\delta$, the reverse of $\delta$ from $b_1$ to $u''$ and finally $\{u'', u'\}$.

Then one sees from the location of $\gamma$ and $\delta$ that $J_1 \supset (0, n)^2$. It also follows from the fact that $\rho_{4k-1}$ is a left-right crossing of $[-n, n] \times [0, n]$, that $\rho_{4k-1}$ contains a piece $\rho'_{4k-1}$ which connects a point $c_{4k-1}$ on $\delta$ to the segment from $a_1$ to $(n, n)$ on the right edge of $S(n)$. In fact we shall take for $\rho'_{4k-1}$ the piece of $\rho_{4k-1}$ from its last intersection with $\delta$ (when starting on $\{n\} \times [0, n]$) to its endpoint $d_{4k-1}$ on $\{n\} \times [-n, n]$. Clearly $\rho'_{4k-1}$ minus its endpoints is contained in $\hat{S}(n)$. Thus by (4.14) (with $J$ taken as the boundary of $S(n)$, viewed as a Jordan curve, and $\tau = \delta$ the reverse of $\delta$, followed by $\{u'', u'\}$ and $\gamma$), we see that

$$\rho'_{4k-1}$$ is a crosscut of $\hat{J}_1$ \hfill (4.15)

(see Figure 4). For the same reason, for $i = 1, 2, 3$, $\tau_i$ contains a piece $\tau'_i$ from a point $e_i$ on $\gamma$ to a point $f_i$ of $s$ on the top edge of $S(n)$, such that $\tau'_i$ is a crosscut of $\hat{J}_1$. In particular the piece of $\tau_i$ between $e_i$ and $f_i$ does not intersect $\gamma$ except at $e_i$ (see Figure 4 again).
Now, by (4.15), \( \rho'_{4k-1} \) divides \( \hat{J}_1 \) into two components, which we denote by \( K^\gamma \) and \( K^s \). The boundary of \( K^\gamma \) consists of \( \gamma \), followed by the segment of \( \{ n \} \times [-n, n] \) from \( a_1 \) to \( d_{4k-1} \) along the right edge of \( S(n) \), then by \( \rho'_{4k-1} \) to \( c_{4k-1} \) and by the piece of \( \delta \) from \( c_{4k-1} \) to \( u'' \) and finally by \( \{ u'', u' \} \). \( K^s \) has \( s \) in its boundary. \( \tau'_i \) starts on \( \gamma \) which is in the boundary of \( K^\gamma \) and ends on \( s \), which is contained in the boundary of \( K^s \). Thus the crosscut \( \tau'_i \) of \( \hat{J}_1 \) runs from one component of \( \hat{J}_1 \setminus \rho'_{4k-1} \) to the other. Consequently, \( \tau'_i \) must have a first intersection \( g_i \) with \( \rho'_{4k-1} \) (when starting at \( e_i \)). Call the piece of \( \tau'_i \) from \( e_i \) to \( g_i \), \( \tau''_i \). Again by (4.14) (with \( J_1 \) taken for \( J \)),

\[
\tau''_i \text{ is a crosscut of } K^\gamma, \ i = 1, 2, 3. \tag{4.16}
\]

Hereafter the construction of the snake will take place in the component \( K \) of \( K^\gamma \) whose boundary consists of the following pieces: \( \gamma \) from \( u' = 0 \) to \( e_3 \), \( \tau''_3 \) (which runs from \( e_3 \) to \( g_3 \)), the piece of \( \rho'_{4k-1} \) from \( g_3 \) to the endpoint \( c_{4k-1} \) of \( \rho'_{4k-1} \) on \( \delta \), the piece of \( \delta \) from this endpoint to \( u'' \), and \( \{ u'', u' \} \). We denote this boundary (when viewed as a Jordan curve) by \( J_2 \). For later use we point out that the first two pieces of \( J_2 \) (i.e., the piece of \( \gamma \) and \( \tau''_3 \)) are occupied while the pieces of \( \rho'_{4k-1} \) and \( \delta \) are vacant.

**Step 3.** Here we exhibit some crosscuts of \( K \) which will be used in the construction of our snake. We claim that

\[
\text{the piece } \tau''_i \text{ of } \tau'_i \text{ between } e_i \text{ and } g_i \text{ is a crosscut of } K, \ i = 1, 2. \tag{4.17}
\]

This is a consequence of the locations of the \( \tau_i \) and (4.9), as we shall now demonstrate. The fact that \( \tau''_3 ([3n/4, n] \times [-n, n]) \subset \tau''_2 ([3n/4, n] \times [-n, n]) \) implies that if we move along the top edge of \( S(n) \) from \( (n, n) \) to \( b_1 \), then we must meet \( f_3 \) before \( f_2 \) (because in the neighborhood of \( f_3 \) there are points which lie in \( \tau''_3 ([3n/4, n] \times [-n, n]) \) and hence in \( \tau''_2 ([3n/4, n] \times [-n, n]) \); thus when we reach \( f_3 \) we cannot have left \( \tau''_2 ([3n/4, n] \times [-n, n]) \) yet). This says that \( f_2 \) lies on the segment of \( s \) between \( f_3 \) and \( b_1 \). Near \( f_2 \) there are therefore points of \( \tau''_2 \) which lie in the component of \( \hat{J}_1 \setminus \tau''_3 \) whose boundary consists of the piece of \( s \) from \( f_3 \) to \( b_1 \), (the reverse of) \( \delta \), \( \{ u'', u' \} \), the piece of \( \gamma \) from \( u' \) to \( e_3 \) and \( \tau'_3 \). Since \( \tau''_2 \) does not intersect the boundary of this component as one moves from \( e_2 \) to \( f_2 \), also \( e_2 \) lies in the part of \( \gamma \) which belongs to the boundary of this component, i.e., the part of \( \gamma \) between \( u' \) and \( e_3 \). But this part of \( \gamma \) also belongs to the boundary of \( K \), so that \( e_2 \) lies in the boundary of \( K \). Then \( \tau''_2 \) must be a crosscut of \( K \) (since we already know from (4.16) that \( \tau''_2 \) either lies entirely in \( K \) or entirely in the other component of \( K^\gamma \setminus \tau''_3 \)). The same argument can be made with the subscript 1 replacing the subscript 2, if we take into account that one meets \( f_3 \) before \( f_1 \) as one moves along \( s \) from \( (n, n) \) to \( b_1 \). Indeed \( \tau_1 \) and \( f_3 \) lie to the right of the vertical line \( x = 3n/4 \) and \( \tau_1 \) and \( f_1 \) lie to the left of the vertical line \( x = n/4 \). This establishes our claim (4.17).

Since the \( \tau_i \) are top-bottom crossings of \( S(n) \) we shall think of \( \tau''_1 \) and \( \tau''_2 \) as “vertical crossings” of \( K \). This terminology is merely a crutch for us to form some mental picture of the \( \tau''_i \) and also to distinguish them from the “horizontal crossings” of \( K \) which we now construct from the \( \rho_j \), \( 1 \leq j \leq 4k - 2 \). In fact, it follows from the fact that \( \rho_j \) is a left-right crossing of \( [-n, n] \times [0, n] \), that \( \rho_j \) must intersect \( \delta \) as well as \( \tau''_3 \). Let \( c_j \) be the last intersection of \( \rho_j \) with \( \delta \) and let \( h_j \) be the first
intersection of $\rho_j$ and $\tau_3''$ after $c_j$, as one moves from $\{-n\} \times [0,n]$ to $\{n\} \times [0,n]$ along $\rho_j$. Finally, let $d_j$ be the endpoint of $\rho_j$ on $\{n\} \times [0,n]$ (which is part of the right edge of $S(n)$). As in (4.15), the piece of $\rho_j$ between $c_j$ and $d_j$ is a crosscut of $J_1$. The same argument as used for (4.17) now shows that for each $j \leq 4k-2$, the piece $\rho_j''$ of $\rho_j$ from $c_j$ to $h_j$ is a crosscut of $K$. The role of the relation (4.9) in the argument will now be taken over by the relation

$$\rho_{4k-1}^+(S(n)) \subset \rho_j^+(S(n)),$$

which follows from (4.4). In fact, $K \setminus \rho_j''$ consists of two components, which we shall denote by $K_j^\pm$, where $K_j^+$ ($K_j^-$) contains a piece of $\rho_{4k-1}'$ (a piece of $\gamma$, respectively) in its boundary (see Figure 5). Then the argument which shows that $\rho_j''$ is a crosscut of $K$ shows at the same time that

$$K_{j+1}^+ \subset K_j^+, \quad 1 \leq j \leq 4k-2. \quad (4.18)$$

The intuitive picture is now as indicated in Figure 5. We have the component $K$, which is “close to” the square $[0,n]^2$, and whose boundary consists of an occupied path on $M$ from $u'$ to a neighbor of $g_3$ (which we shall call $v'$), and a vacant path from $u''$ to $g_3$; $g_3$ will be $v''$. (Note that $g_3$ is necessarily a central vertex, for otherwise it would have to be vacant, as a vertex on $\rho_{4k-1}$, as well as occupied, as a vertex of $\tau_3$. We therefore can take $v'' = g_3$ without having to check that $g_3$ is vacant.) To the left of $x = n/4$ (right of $3n/4$) we have an occupied (vacant) vertical crossing $\tau_1''$ ($\tau_2''$, respectively). We also have a sequence of horizontal crossings $\rho_j''$ with $\rho_{j+1}''$ “above” $\rho_j''$.

![Figure 5](image)

Figure 5. The component $K$ with the crossings $\tau_1'', \tau_2''$ and some $\rho_j''$.

**Step 4.** We now define further paths $\sigma_\ell$, pieces of which will go into the boundary of the snake. We shall use the notation

$$A^{cl} = \text{(topological) closure of } A$$

for $A \subset \mathbb{R}^2$. 
The definition of $\sigma_\ell$ depends on the parity of $\ell$. Let us first consider the case when $\ell$ is odd. Then note that $\tau''_1$ runs between $e_1$ and $g_1$. These points lie in different components of $K \setminus \rho''_3$; $e_1 \in K^-_j$ and $g_1 \in K^+_j$. Therefore, as one traverses $\tau''_1$ from $e_1$ to $g_1$ there is a last point in the $(K^-_{2\ell-1})^c$. Call this point, which is an intersection of $\tau''_1$ and $\rho_{2\ell-1}$, $y_{2\ell-1}$. After this point $\tau''_1$ lies in $K^+_{2\ell-1}$, but in some neighborhood of $y_{2\ell-1}$, $\tau''_1$ is still in $K^-_{2\ell}$. The first point after $y_{2\ell-1}$ where $\tau''_1$ hits $\rho''_2$, we shall denote by $z_{2\ell}$. Thus, the piece of $\tau''_1$ from $y_{2\ell-1}$ to $z_{2\ell}$ (minus its endpoints $y_{2\ell-1}$ and $z_{2\ell}$) $\subset K^-_{2\ell} \cap K^+_{2\ell-1}$. (4.19)

We now define $\sigma_\ell$ as the self-avoiding path which consists of the concatenation of the following three paths: the piece of $\rho''_{2\ell-1}$ from $h_{2\ell-1}$ to $y_{2\ell-1}$, the piece of $\tau''_1$ from $y_{2\ell-1}$ to $z_{2\ell}$ and the piece of $\rho''_2$ from $z_{2\ell}$ to $h_{2\ell}$. For odd $\ell$ all these pieces are occupied (by (4.3) and (4.8)) so that

$$\sigma_\ell \text{ is occupied for odd } \ell.$$ (4.20)

The endpoints $h_{2\ell-1}$ and $h_{2\ell}$ of $\sigma_\ell$ lie on the boundary of $K$ and $\sigma_\ell$ contains points of $\tau''_1$ which lie in $K$. Thus,

$$\sigma_\ell \text{ is a crosscut of } K.$$ (4.21)

For $\ell$ even we interchange the roles of left and right. We now define $\sigma_\ell$ as a path consisting of pieces of $\rho_{2\ell-1}$ and $\rho_{2\ell}$ which run from $c_{2\ell-1}$ to $\tau''_1$ and from $\tau''_2$ to $c_{2\ell}$ with a connecting piece of $\tau''_2$. We will again have (4.21), but

$$\sigma_\ell \text{ is vacant for even } \ell.$$ (4.22)

For odd $\ell$ we shall denote by $H_\ell$ that component of $K \setminus \sigma_\ell$ whose boundary consists of $\sigma_\ell$ and the piece of $\tau''_3$ between $h_{2\ell-1}$ and $h_{2\ell}$ (but no pieces of $\rho_{4k-1}$, $\delta$ or $\gamma$). Note that this boundary lies in

$$(K^+_{2\ell-1})^c \cap (K^-_{2\ell})^c.$$ (4.23)

In fact the parts of the boundary of $H_\ell$ on $\tau''_1$ and on $\rho_{2\ell}$ together also form a crosscut of $K^+_{2\ell-1}$ and an alternative description of $H_\ell$ is therefore that

$H_\ell$ is the component of $K^+_{2\ell-1} \setminus \sigma_\ell$ whose boundary consists of $\sigma_\ell$ and the piece of $\tau''_3$ between $h_{2\ell-1}$ and $h_{2\ell}$.

(4.24)

Since the boundary of $H_\ell$ lies in the set (4.23) we also have (see Newman (1951), Theorem V.11. 1 and its proof)

$$H_\ell \subset K^+_{2\ell-1} \cap K^-_{2\ell}.$$ (4.25)

For even $\ell$ we take for $H_\ell$ that component of $K \setminus \sigma_\ell$ whose boundary consists of $\sigma_\ell$ and the piece of $\delta$ between $c_{2\ell-1}$ and $c_{2\ell}$ (but no pieces of $\rho_{4k-1}$, $\tau''_3$ or $\gamma$). Then (4.25) also holds for even $\ell$. 
Step 5. We shall now define a path $\zeta$, which will in fact be the boundary of our snake. Roughly speaking, $\zeta$ is the Jordan curve $J_2$, which is the boundary of $\mathcal{K}$, except that the pieces between $h_{2\ell-1}$ and $h_{2\ell}$ are replaced by $\sigma_\ell$ for $\ell$ odd, and the pieces between $c_{2\ell-1}$ and $c_{2\ell}$ are replaced by $\sigma_\ell$ for $\ell$ even. Somewhat more formally, to traverse $\zeta$ we start at $u' = 0$ and move along $\gamma$ to $e_3$. We then move along $\tau''_3$ from $e_3$ to $h_1$. From $h_1$ to $h_2$ we do not move along $\tau''_3$, but instead follow $\sigma_1$. After arriving at $h_2$ we continue along the piece of $\tau''_3$ between $h_2$ and $h_5$. We then move from $h_5$ to $h_6$ along $\sigma_2$. We continue in this way till we reach $h_{4k-2}$. From there we go along $\tau''_3$ till its intersection with $\rho''_{4k-1}$ (which we called $g_3$). The part of $\zeta$ described so far is occupied. We now go back to $u''$ by a vacant piece of $\zeta$. This part of $\zeta$ consists first of $\rho''_{4k-1}$ from $g_3$ to $c_{4k-1}$, the intersection of $\rho''_{4k-1}$ with $\delta$. We then move along the reverse of $\delta$ to $c_{4k-4}$. Instead of following the piece of $\delta$ from $c_{4k-4}$ to $c_{4k-5}$ we follow the reverse of $\sigma_{2k-2}$ from $c_{4k-4}$ to $c_{4k-5}$. We then go along the reverse of $\delta$ to $c_{4k-8}$ and follow the reverse of $\sigma_{2k-4}$ etc., until we arrive at $c_3$. From there we move along the reverse of $\delta$ to $u''$. Finally, to make $\zeta$ into a closed curve we add the segment between the adjacent points (on $\mathcal{M}$) $u''$ and $u'$.

We claim that $\zeta$ is a Jordan curve, located in the closure of $\mathcal{K}$. This follows quickly from the construction of the $\sigma_\ell$. Each $\sigma_\ell$ is a self-avoiding path which lies in $\mathcal{K}$ except for its endpoints $h_{2\ell-1}$, $h_{2\ell}$ or $c_{2\ell-1}$, $c_{2\ell}$, which lie on the boundary of $\mathcal{K}$, so that indeed $\zeta \subset (\mathcal{K})^{cl}$. We therefore only have to prove that the different $\sigma_\ell$ are disjoint. But we already saw that $\sigma_{4k-2}$ is contained in the set (4.23) and we claim that these regions are disjoint for different $\ell$. Indeed

$$
(K_p^+)^{cl} \cap (K_p^-)^{cl} = \rho''_p
$$

(4.26)

essentially by definition (see Newman (1951), Theorem V.11.8). Moreover, for $j > p$,

$$
\rho''_p \cap (K_j^+)^{cl} = \emptyset.
$$

(4.27)

To see this note that $\rho''_p$ cannot contain any point of $K_j^+$, because in the neighborhood of such a point there would be points outside $K_p^+$, and hence outside $K_j^+$ (by (4.18)). Neither can $\rho''_p$ intersect $\rho''_j$, by assumption (4.2). Therefore $\rho''_p \cap (K_j^+)^{cl}$ can only consist of points in (boundary of $K_j^+ \setminus \rho''_j$. This is an arc of $J_2$, the boundary of $\mathcal{K}$. But if there were a point in $\rho''_p \cap (\text{boundary of } K_j^+) \setminus \rho''_j$, then again any neighborhood of that point would contain points outside $K_p^+$ but inside $K_j^+$. As we just saw this is impossible, so that (4.26) holds. Finally, (4.18), (4.26) and (4.27) together show that

$$
(K_j^+)^{cl} \cap (K_p^-)^{cl} \subset (K_j^+)^{cl} \cap (K_p^+)^{cl} \cap (K_p^-)^{cl} = \emptyset \text{ for } j > p.
$$

(4.28)

Thus the regions in (4.23) and the $\sigma_\ell$ for different $\ell$ are indeed disjoint.

Step 6. In this step we complete the proof of the lemma. Let $\zeta^\circ$ denote the interior of $\zeta$. Since $\zeta \subset (\mathcal{K})^{cl}$, it must be the case that $\zeta^\circ \subset \mathcal{K}$ (see Newman (1951), proof of Theorem V.11.1), and

$$
(\zeta^\circ)^{cl} = \zeta^\circ \cup \zeta \subset (\mathcal{K})^{cl}.
$$
Now let $\phi : [0, 1] \to \zeta \cup \zeta$ be a path in $\zeta \cup \zeta$ which starts at $u' = 0$ and ends at or adjacent to $g_3$. Then $\phi$ is also a path in the closure of $\mathcal{K}$. It begins at $u' \in (\mathcal{K}_\downarrow)^{cl} \subset (\mathcal{K}_\downarrow^-)^{cl}$ for all $j$, and ends near $g_3$ and hence in $(\mathcal{K}_\downarrow^+)^{cl} \subset (\mathcal{K}_j^+)^{cl} \setminus \rho_j''$ for all $j \leq 4k - 2$ (note that $g_3 \not\in \rho_j$ for $j \leq 4k - 2$). $\phi$ must therefore intersect each of the $\rho_j''$ which separate $\mathcal{K}_j^+$ from $\mathcal{K}_j^-$. In fact more is true. If $w_p$ is the last point of $\phi$ on $\rho_p$, then $\phi$ must still intersect $\rho_j''$ for all $j > p$ after $w_p$ (because also $w_p \in \rho_p'' \subset (\mathcal{K}_p^-)^{cl} \subset \mathcal{K}_j^-$). For $j = 2q - 1$ with $q$ odd, denote the part of $\rho_j''$ between $y_{2q-1}$ (on $\tau_1''$) and $c_{2q-1}$ (on $\delta$) by $\lambda_{2q-1}$ (see Figure 6). When $q$ is even

![Figure 6](image)

**Figure 6.** A path $\phi$ (solidly drawn) inside the “snake”.

let $\lambda_{2q-1}$ be the piece of $\rho_{2q-1}$ between its first intersection with $\tau_2''$ (when starting at $c_{2q-1}$) and $h_{2q-1}$.

The most important part of our argument is now that

$$
\phi \text{ must intersect } \lambda_{2q-1} \text{ after } w_p \text{ for } 2q - 1 > p. \quad (4.29)
$$

For the sake of argument we prove this for odd $q$. We first observe that the boundary of each $\mathcal{H}_q$ is disjoint from $\zeta$. Indeed, this boundary consists of $\sigma_q \subset \zeta$, which certainly lies outside $\zeta$, and the piece of $\tau_3''$ strictly between $h_{2q-1}$ and $h_{2q}$. The latter piece is disjoint from $\zeta$, because this piece was replaced by $\sigma_q$ in the construction of $\zeta$. But then this piece of $\tau_3''$ either lies entirely in $\zeta$ or entirely in the exterior of $\zeta$. Because points of $\tau_3''$ can be connected to infinity by paths which lie outside the closure of $\mathcal{K}$ (except for their initial point), and hence in the exterior of $\zeta$, the piece of $\tau_3''$ between $h_{2q-1}$ and $h_{2q}$ lies in the exterior of $\zeta$. This proves our claim that the boundary of $\mathcal{H}_q$ is disjoint from $\zeta$.

The fact that the boundary of $\mathcal{H}_q$ is disjoint from $\zeta$ implies that any two points of $\zeta$ can be connected by a path in $\zeta$ which does not intersect the boundary of $\mathcal{H}_q$. Consequently, all of $\zeta$ either lies in $\mathcal{H}_q$ or in the complement of the $(\mathcal{H}_q)^{cl}$. The latter case must occur, because $\zeta$ contains points arbitrarily close to $u' = 0$ which
can be connected to infinity outside \( \mathcal{H}_q \) (in fact, 0 can be connected to infinity outside \( \mathcal{K} \)). We conclude that \( \hat{\zeta} \cap \mathcal{H}_q = \emptyset \). This of course implies that not even \( \zeta \) can intersect the open set \( \mathcal{H}_q \), so that

\[
(\zeta \cup \zeta) \cap \mathcal{H}_q = \emptyset. \tag{4.30}
\]

Now assume, to derive a contradiction, that (4.29) fails. Since we already know that \( \phi \) must leave \( (\mathcal{K}_{2q-1}^-)^{cl} \) and enter \( (\mathcal{K}_{2q-1}^+)^{cl} \), \( \phi \) must cross \( \rho''_{2q-1} \) after \( w_p \), that is, there must exist a \( t_q \) such that

\[
\phi(t_q) \in \rho''_{2q-1} \text{ and } \phi(t) \in (\mathcal{K}_{2q-1}^+)^{cl} \setminus \rho''_{2q-1}
\]

for some \( t > t_q \), arbitrarily close to \( t_q \). \tag{4.31}

If (4.29) fails, then

\[
\phi(t_q) \in \rho''_{2q-1} \setminus \lambda_{2q-1}. \tag{4.32}
\]

This says that \( \phi(t_q) \) lies in the piece of \( \rho''_{2q-1} \) between \( y_{2q-1} \) and \( h_{2q-1} \), that is, the piece of \( \rho''_{2q-1} \) in the boundary of \( \mathcal{H}_q \). Also, \( \phi(t_q) \neq y_{2q-1} \in \lambda_{2q-1} \). But, as we saw in the lines before (4.25), the parts of \( \sigma_q \) on \( \rho''_{2q-1} \) and on \( \tau''_1 \) form a crosscut of \( \mathcal{K}_{2q-1}^+ \). \( \mathcal{H}_q \) is one of the components of \( \mathcal{K}_{2q-1}^+ \) after this crosscut is removed from \( \mathcal{K}_{2q-1}^+ \), and (4.29) puts \( \phi(t_q) \) in the boundary of \( \mathcal{H}_q \), but not in boundary of the other component of \( (\mathcal{K}_{2q-1}^+ \text{ minus the crosscut}) \). Therefore, \( \phi(t_q) \) has some neighborhood \( U \) so that

\[
U \cap \mathcal{K}_{2q-1}^+ \subset \mathcal{H}_q. \tag{4.33}
\]

But, by virtue of \( \phi(t) \in \hat{\zeta} \cup \zeta \) and of (4.30), we cannot have \( \phi(t) \in \mathcal{H}_q \). Thus for some \( t > t_q \), but arbitrarily close to \( t_q \) it must be the case that

\[
\phi(t) \in (\mathcal{K}_{2q-1}^+)^{cl} \setminus (\rho''_{2q-1} \cup \mathcal{K}_{2q-1}^+). \tag{4.34}
\]

(see (4.31) and (4.33)). Since \( \mathcal{K}_{2q-1}^+ \) is a component of \( \mathcal{K} \setminus \rho''_{2q-1} \), the right hand side here is contained in the arc of the boundary of \( \mathcal{K} \) from \( h_{2q-1} \) to \( c_{2q-1} \) which contains \( \rho''_{4k-1} \). But as \( t \downarrow t_q \), the points \( \phi(t) \) here have to approach \( \phi(t_q) \in \rho''_{2q-1} \). By our assumption, \( \phi(t_q) \notin \lambda_q \), so that in particular, \( \phi(t_q) \neq c_{2q-1} \). This forces \( \phi(t) \) to take values on the segment of \( \tau''_1 \) strictly between \( h_{2q-1} \) and \( h_{2q} \). This, however, is also impossible, because we already proved in the lines following (4.29) that this arc lies in the exterior of \( \zeta \). Thus (4.29) must hold.

It is now easy to complete the proof. (4.29) also holds when \( j = 2q - 1 \) for an even \( q \), by the same proof as for odd \( q \), with only the roles of left and right interchanged. Now (4.29) implies that a path \( \phi \) from \( u' \) to \( \{u', v''\} \) must successively intersect \( \lambda_1, \lambda_3, \ldots, \lambda_{2k-1} \). But \( \tau''_1 \) lies to the left of the vertical line \( x = n/4 \). Thus as one traverses \( \rho''_{2q-1} \) from \( h_{2q-1} \) to \( c_{2q-1} \) one first hits the vertical line \( x = n/4 \) before one hits \( \tau''_1 \) and also the point \( y_{2q-1} \) of \( \tau''_1 \) must lie on or to the left of the line \( x = n/4 \). Since \( \rho''_1 \) has only one point in common with the vertical line \( x = n/4 \) (by assumption (4.4)), the piece \( \lambda_{2q-1} \) from \( y_{2q-1} \) to \( c_{2q-1} \) must lie entirely in the half plane \( (-\infty, n/4] \times \mathbb{R} \). For similar reasons, for even \( q \), \( \lambda_{2q-1} \subset [3n/4, \infty] \times \mathbb{R} \). Thus, the distance between \( \lambda_{2r-1} \) and \( \lambda_{2r+1} \) is at least \( n/2 \), for each \( 1 \leq r \leq 2k - 2 \).
Consequently, if \( \phi \) successively intersects \( \lambda_1, \lambda_3, \ldots, \lambda_{4k-3} \), then its length is at least \((2k - 2)n/2\). \( \square \)

The remainder of the proof uses fairly standard block techniques from percolation to show that the conditions of Lemma 2 with \( k \sim [cn] \) are fulfilled with a probability which rapidly approaches 1 as \( n \to \infty \). First we introduce our blocks or renormalized sites. The renormalized site \((i,j)\) will depend on the configuration of occupied and vacant sites in the square \([ (14i - 13)N, (14i + 13)N - 1 ] \times [ (14j - 13)N, (14j + 13)N - 1 ]\) for an \( N \) to be determined below. The renormalized site \((0,0)\) will be colored white if all the paths listed in (4.35)-(4.40) exist on \( \mathcal{M} \). The reader is advised to look at Figure 7 for these paths, before reading the formal description.

there exists an occupied circuit surrounding the origin  
in each of the 2 annuli \( S(4N) \setminus S(3N), S(5N) \setminus S(4N) \).  
(4.35)

there exists a vacant circuit surrounding the origin  
in each of the 4 annuli \( S(2N) \setminus S(N), S(3N) \setminus S(2N) \), \( S(6N) \setminus S(5N) \) and \( S(7N) \setminus S(6N) \);  
(4.36)

there exist occupied left-right crossings of each of the 4 rectangles  
\([-13N, -N + 1] \times [-N/3, -1], [-13N, -N + 1] \times [0, N/3 - 1], \]
\([N, 13N - 1] \times [-N/3, -1] \) and \([N, 13N - 1] \times [0, N/3 - 1] \);  
(4.37)

there exist occupied top-bottom crossings of each of the 4 rectangles  
\([-N/3, -1] \times [N, 13N - 1], [0, N/3 - 1] \times [N, 13N - 1], \]
\([-N/3, -1] \times [-13N, -N + 1] \) and \([0, N/3 - 1] \times [-13N, -N + 1] \);  
(4.38)

there exist vacant left-right crossings of each of the 8 rectangles  
\([-13N, -N + 1] \times [-N, -2N/3 - 1], [-13N, -N + 1] \times [-2N/3, -N/3 - 1], \]
\([-13N, -N + 1] \times [N/3, 2N/3 - 1], [-13N, -N + 1] \times [2N/3, N], \]
\([N, 13N - 1] \times [-N, -2N/3 - 1], [N, 13N - 1] \times [-2N/3, -N/3 - 1], \]
\([N, 13N - 1] \times [N/3, 2N/3 - 1] \) and \([N, 13N - 1] \times [2N/3, N] \);  
(4.39)
there exist vacant top-bottom crossings of each of the 8 rectangles
\[\begin{align*}
[-N, -2N/3 - 1] \times [N, 13N - 1], & \quad [-2N/3, -N/3 - 1] \times [N, 13N - 1], \\
[N/3, 2N/3 - 1] \times [N, 13N - 1], & \quad [2N/3, N] \times [N, 13N - 1], \\
[-N, -2N/3 - 1] \times [-13N, -N + 1], & \quad [-2N/3, -N/3 - 1] \times [-13N, -N + 1], \\
[N/3, 2N/3 - 1] \times [-13N, -N + 1] \text{ and } [2N/3, N] \times [-13N, -N + 1];
\end{align*}\] (4.40)

The renormalized site \((i, j)\) is colored white if translates by \((14iN, 14jN)\) of the paths in (4.35)-(4.40) exist in \((14iN, 14jN) + [-13N, 13N - 1]^2\). All renormalized sites which are not colored white are colored \textit{black}. Two renormalized sites \((i, j)\) and \((i', j')\) will be adjacent if \(|i - i'| + |j - j'| = 1\), so that the renormalized sites can be viewed as the sites of \(Z^2\). Thus the colorings can be viewed as a site percolation process on \(Z^2\). Since the color of \((i, j)\) depends on the configuration in \([((14i-13)N, (14i+13)N-1] \times [(14j-13)N, (14j+13)N-1] \subset (14iN, 14jN) + S(14N)\) the coloring of the sites is not an not an independent percolation process on \(Z^2\), but a 1-dependent one. That is, the color configurations of two collections of sites \(A_1\) and \(A_2\) are independent if \(|(i, j) - (i', j')|_{\infty} > 1\) for all \((i, j) \in A_1, (i', j') \in A_2\).

It is clear that
\[\alpha_N := P_p\{(i, j) \text{ is white}\} \text{ is independent of } (i, j).\] (4.41)

It is also standard that
\[\lim_{N \to \infty} \alpha_N = 1,\] (4.42)
because for our percolation process on the original lattice $\mathbb{Z}_{cp}^2$, under (1.2),

$$
P_p \{ \# \text{ occupied left-right crossing of } [0, [xN]] \times [0, N] \text{ on } \mathbb{Z}_{cp}^2 \} \\
= P_p \{ \exists \text{ vacant path on } \mathbb{Z}^2 \text{ from } [0, [xN]] \times \{1\} \text{ to } [0, [xN]] \times \{N - 1\} \\
in [0, [Nx]] \times [1, N - 1] \}
\leq C_7 \exp(-C_8N)
$$

(4.43)

for some constants $C_i \in (0, \infty)$ which depend on $p$ and $x$ only (see Kesten (1982), Proposition 2.2 (corrected to replace $J \setminus A_1 \cup A_3$ by $\overline{J} \setminus A_1 \cup A_3$ on p. 30, L. 1, 2 f.b.) and Theorem 5.1; see also Grimmett (1999), p.293 and Theorem 5.4). The same statement is true for vacant crossings and for occupied and vacant top-bottom crossings of $[0, N] \times [0, [xN]]$. These exponential bounds for the non-existence of paths on $\mathbb{Z}_{cp}^2$ immediately imply similar bounds on $M$. An exponential upper bound for the nonexistence of circuits in the annuli $S((i + 1)N) \setminus S(iN)$, $1 \leq i \leq 6$, also follows from (4.43) (see Grimmett (1999), Theorem 11.70).

Our definition of the coloring also has the following geometric (deterministic) consequence: Assume that

there is a white path $v_1, v_2, \ldots, v_r$, with $v_k = (i_k, j_k)$

which forms a left-right crossing of $[a, b] \times [c, d]$ on $\mathbb{Z}^2$.

(4.44)

Then

there exist on $M$ two occupied left-right crossings, $\rho_1, \rho_2$

and two vacant left-right crossings, $\rho_3, \rho_4$, of

$$
in the set $\Lambda(v_1, \ldots, v_r) := \bigcup_{k \leq r} (14v_k + [-7N, 7N - 1] \times [7N, 7N - 1])$

$\cup [(14i_1 - 13)N, (14i_1 - 7)N - 1] \times [(14j_1 - 7)N, (14j_1 + 7)N - 1]$

$\cup [(14i_r + 7)N, (14i_r + 13)N - 1] \times [(14j_r - 7)N, (14j_r + 7)N - 1]$

(4.45)

(note that $i_1 = a, i_r = b$); also

$$
\rho_1 - \rho_4 \text{ will be disjoint;}
$$

(4.46)

in addition these crossings will be such that

$$
\rho_1 \text{ and } \rho_2 \text{ lie “below” } \rho_3, \rho_4.
$$

(4.47)

If we write $\rho_i^+ = \rho_i^+(a, b, c, d)$ for the component of

$$
[(14a - 13)N, (14b + 13)N - 1] \times [(14c - 13)N, (14d + 13)N - 1] \setminus \rho_i,
$$

then the precise meaning of (4.47) is that

$$
\rho_4^+ \subset \rho_3^+ \subset \rho_2^+ \subset \rho_1^+.
$$

(4.48)
If \( ((14a - 13)N, \ell_i) \) and \( ((14b + 13)N - 1, r_i) \) are the endpoints of \( \rho_i \) on the left and right edge of \( [(14a - 13)N, (14b + 13)N - 1] \times [(14c - 7)N, (14d + 7)N - 1] \), respectively, then an equivalent way of expressing (4.46) (under (4.34)) is that

\[
\ell_1 < \ell_2 < \ell_3 < \ell_4,
\]

or that

\[
r_1 < r_2 < r_3 < r_4.
\]

We leave a formal proof of this to the reader. We merely illustrate in Figure 8 how two disjoint vacant and two disjoint occupied left-right crossings of \( [(14i - 7)N, (14i - 1)N - 1] \times [(14j - 1)N, (14j + 1)N] \) can be continued through \( [(14i - 7)N, (14i + 7)N - 1] \) and end in top-bottom crossings of \( [(14i - 1)N, (14i + 1)N] \times [(14j - 13)N, (14j - 1)N - 1] \), or of \( [(14i - 1)N, (14i + 1)N] \times [(14j + 1)N, (14j + 13)N - 1] \), provided the renormalized site \( (i, j) \) is white.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{Two illustrations of the continuation of two disjoint occupied and two disjoint vacant paths through a white site.}
\end{figure}

We now show that the properties listed above are enough to give us (1.5). The following lemma is the main step.

**Lemma 3.** There exists an \( N \) and constants \( C_i = C_i(p) \in (0, \infty) \) such that for all large \( n \),

\[
P_p\{(4.1)-(4.5) \text{ of Lemma 2 with } n \text{ replaced by } 56Nn \text{ and with } k = C_{9n} \text{ hold}\} \\
\geq 1 - C_{10} \exp(-C_{11}n).
\]

**Proof.** Define the following events for the renormalized sites:

\[F_1 = \{\text{there exist at least } 3n \text{ disjoint white left-right crossings of } [-4n, n - 2] \times [1, 4n - 1] \};\]

\[F_2 = \{\text{there exist at least } 3n \text{ disjoint white left-right crossings of } [n + 2, 3n - 2] \times [1, 4n - 1] \};\]
and

\[ F_3 = \{ \text{there exist at least } 3n \text{ disjoint white left-right crossings of } [3n+2, 4n] \times [1, 4n-1] \}. \]

Assume that \( F_1 \) occurs. Then there exist integers \( a_1 < a_2 < \cdots < a_{3n} \subset [1, 4n-1] \) and \( 3n \) disjoint white left-right crossings of \([-4n, n-2] \times [1, 4n-1] \) whose endpoints on \( \{(n-2) \times [1, 4n-1]\) are \((n-2, a_1), (n-2, a_2), \ldots, (n-2, a_{3n})\). Similarly, if also \( F_2 \) occurs, then there are \( 3n \) disjoint white left-right crossings of \([n+2, 3n-2] \times [1, 4n-1] \) with left and right endpoints \((n+2, b_i)\) and \((3n-2, c_i)\), respectively, with \( 1 \leq b_1 < b_2 < \cdots < b_{3n} \leq 4n-1 \) and \( 1 \leq c_1 < c_2 < \cdots < c_{3n} \leq 4n-1 \). Finally, if \( F_3 \) occurs, then there exist \( 3n \) disjoint white left-right crossings of \([3n+2, 4n] \) with left endpoints \((3n+2, d_i)\) with \( 1 \leq d_1 < \cdots < d_{3n} \leq 4n-1 \).

Now let \( F_1 \cap F_2 \cap F_3 \) occur and let \( a_j, b_i, c_i, d_i \) be as in the preceding paragraph. The number of integers in \([1, 4n-1]\) which are not equal to one of the \( a_j \) is at most \( n-1 \). Thus, at least \( 3n - (n-1) \geq 2n \) of the \( b_i \) are also equal to some \( a_j \). For similar reasons, among 2n of the \( i \) for which \( b_i \) equals some \( a_j \), there are at least \( n \) values of \( i \) for which \( c_i \) equals some \( d_j \). Thus, there exist at least \( n \) pairs \((b_i, c_i)\) such that \( b_i \) equals some \( a_j \) and \( c_i \) equals some \( d_j \). By discarding some of the \( a_i - d_i \) and renumbering the remaining ones, we therefore can find \( b_1 < \cdots < b_n \) and \( c_1 < \cdots < c_n \) for which there exist disjoint white left-right crossings of \([-4n, n-2] \times [1, 4n-1] \) with right endpoints \((n-2, b_i)\), disjoint white left-right crossings of \([n+2, 3n-2] \times [1, 4n-1] \) from \((n+2, b_i)\) to \((3n-2, c_i)\) and disjoint white left-right crossings of \([3n+2, 4n] \times [1, 4n-1] \) with left endpoints \((3n+2, c_i)\). By properties (4.45)-(4.47) this means that there exist for each \( 1 \leq i \leq n \) on \( \mathbb{Z}_p^2 \) two occupied left-right crossings \( \rho_{i, 1}, \rho_{i, 2} \) and two vacant left-right crossings \( \rho_{i, 3}, \rho_{i, 4} \) of

\[
([-4n(14) - 13)N, ((n - 2)(14) + 13)N - 1] \times [0, 4n(14)N]
\]

whose right endpoint lies in \( \{(n - 2)(14) + 13)N - 1 \} \times (b_i(14) - 7)N, (b_i(14) + 7)N - 1 \), and all four of them disjoint. Similarly, the white path from \((n+2, b_i)\) to \((3n-2, c_i)\) gives us two occupied left-right crossings \( \rho'_{i, 1}, \rho'_{i, 2} \) and two vacant left-right crossings \( \rho'_{i, 3}, \rho'_{i, 4} \) of \([-4n(14) + 13)N, (3n-2)(14)N - 1] \times [0, 4n(14)N] \) with starting point in \( \{(n + 2)(14) - 13)N \} \times (b_i(14) - 7)N - N, (b_i(14) + 7)N - 1 \) and endpoint in \( \{(3n-2)(14)N - 1 \} \times (c_i(14) - 7)N, (c_i(14) + 7)N - 1 \). All four of these paths are disjoint. Finally, there exist similar left-right crossings \( \rho''_{i, 1}, \rho''_{i, 2} \) with starting points in \( \{(3n+2)(14) - 13)N \} \times (c_i(14) - 7)N, (c_i(14) + 7)N - 1 \). Moreover, \( \rho_{i, 1} \) and \( \rho_{i, 2} \) lie below \( \rho_{i, 3} \) and \( \rho_{i, 4} \) and the same statement holds for the \( \rho' \) and for the \( \rho'' \).

We now want to connect \( \rho_{i, j}, \rho'_{i, j}, \rho''_{i, j} \) to form a left-right crossing of the whole rectangle \([-4n(14) - 13)N, (4n(14) + 13)N - 1] \times [0, 4n(14)N] \), in such a way that the resulting family of crossings (when suitably renumbered) satisfies (4.1)-(4.5). It is clear that there exists a path \( \psi_{i,j} \) on \( \mathcal{M} \) in

\[
([n(14) + 13)N, (n+2)(14) - 13)N] \times (b_i(14) - 7)N, (b_i(14) + 7)N - 1 \]

which connects \( \rho_{i, j} \) to \( \rho'_{i, j} \). It is even possible to choose \( \psi_{i,j} \) such that it intersects the vertical line \( \{x = n(14)N\} \) in one point only. By our construction the order of
the right endpoints of the $\rho_{i,j}$, $1 \leq j \leq 4$, is the same as that of the left endpoints of the $\rho'_{i,j}$ (see (4.49) and (4.50)). It is not hard to see that, for sufficiently large $N$, this allows us to even choose the $\psi_{i,j}$, $1 \leq j \leq 4$, disjoint. In a similar way we can choose disjoint paths $\psi'_{i,j}$, $1 \leq j \leq 4$, on $\mathcal{M}$, inside

$$\left[ \left( (3n-2)(14) + 13 \right) N, (3n+2)(14) - 13 \right] \times \left[ (c_i(14) - 7) N, (c_i(14) + 7) N - 1 \right]$$

(4.52)

which connect $\rho'_{i,j}$ and $\rho''_{i,j}$ in such a way that $\psi'_{i,j}$ intersects the vertical line \{ $x = 3n(14) N$ \} in one point only. We shall denote by by $G(i)$ the event that $\psi_{i,j}$ and $\psi'_{i,j}$ are occupied for $j = 1, 2$ and vacant for $j = 3, 4$. Note now that the occurrence of $F_1 \cap F_2 \cap F_3$ tell us nothing about the occupancy of vertices in

$$\left( \left( (n-2)(14) + 13 \right) N, (n+2)(14) - 13 \right) N - 1 \times \mathbb{Z} \right) \cup \left( \left( (3n-2)(14) + 13 \right) N, (3n+2)(14) - 13 \right) N - 1 \times \mathbb{Z} \right).$$

There exists therefore a $C_{12} > 0$ such that

$$P_p\{G(i)|F_1 \cap F_2 \cap F_3\} \geq C_{12}.$$  

If $F_1 \cap F_2 \cap F_3$ and $G(i)$ occur, then the $\rho_{i,j}, \rho'_{i,j}, \rho''_{i,j}$ with their connections $\psi_{i,j}, \psi'_{i,j}$ form four disjoint left-right crossings of $\left( (4n(14) - 13) N, (4n(14) + 13) N - 1 \right] \times \{0, 4n(14) N\}$, the lower two of which are occupied and the top two of which are vacant. Moreover, these crossings have only one point in common with each of the vertical lines \{ $x = n(14) N$ \} and \{ $x = 3n(14) N$ \}.

Next we note that for given distinct $b_i$ the regions (4.51) are disjoint for different $i$. Similarly, the regions in (4.52) are disjoint. Thus, conditionally on $F_1 \cap F_2 \cap F_3$ and a choice of the $b_i, c_i$, the events $G(i)$, $1 \leq i \leq n$, are independent. Since there are $n$ possible choices for $i$, standard large deviation estimates for the binomial distribution tell us that

$$P_p\{G(i) \text{ occurs for at least } C_6 n/2 \text{ values of } i | F_1 \cap F_2 \cap F_3\} \geq 1 - C_{13} \exp(-C_{14} n).$$

(4.53)

We claim further that the paths which are constructed from the $\rho_{i,j}, \rho'_{i,j}, \rho''_{i,j}, \psi_{i,j}, \psi'_{i,j}$ for different values of $i$ are disjoint. This is so, because for two disjoint paths $v_1, \ldots, v_r$ and $w_1, \ldots, w_s$ on $\mathbb{Z}^2$, the sets $\Lambda(v_1, \ldots, v_r)$ and $\Lambda(w_1, \ldots, w_s)$, in which the paths of (4.45) are located, are disjoint. It follows that if $F_1 \cap F_2 \cap F_3$ occurs, and $G(i)$ occurs for at least $C_{12} n/2$ values of $i$, then we can renumber the resulting left-right crossings of $\left( (4n(14) - 13) N, (4n(14) + 13) N - 1 \right] \times \{0, 4n(14) N\}$ and obtain that (4.1)-(4.5) are satisfied with $n$ replaced by $56 N n$ and $k$ by $C_{12} n/2$. (Note that (4.4) is merely a question of numbering of the crossings, because the components $\rho_{i}^+(S(n))$ are automatically nested if the $\rho_{i}$ are disjoint). In view of (4.53) it therefore suffices for this lemma to show that for large enough $N$

$$P_p\{F_{\ell}\} \geq 1 - C_{15} \exp(-C_{16} n), \quad \ell = 1, 2, 3.$$  

(4.54)

Fortunately it is well known how to prove (4.54) (see Grimmett (1999), Lemma 11.22 and the argument following Theorem 2.45, or Kesten, Theorem 11.1). Since
we do not have independent percolation for the white sites we add a few words. For the sake of argument take \( \ell = 1 \). Note that \( \mathbb{Z}^2 \) and \( \mathbb{Z}_c^2 \) are a matching pair in the terminology of Kesten (1982). Then, as in the references just mentioned, by Menger’s theorem (and Proposition 2.2 in Kesten (1982)), the maximal number of disjoint white left-right crossings of \([-4n, n-2] \times [1, 4n-1]\) on \( \mathbb{Z}^2 \) equals the minimal number of white sites on any self-avoiding path on \( \mathbb{Z}_c^2 \) from \([-4n, n-2] \times \{2\}\) to \([-4n, n-2] \times \{4n-2\}\) inside \([-4n, n-2] \times [2, 4n-2]\). Thus, by a simple Peierls argument

\[
P_p \{ F_1 \text{ fails} \}
\leq P_p \{ \exists \text{ a self-avoiding path on } \mathbb{Z}_c^2 \text{ starting on } [-4n, n-2] \times \{2\} \text{ and containing } 4n - 3 \text{ sites, but with fewer than } 3n \text{ white sites} \}
\leq (5n - 1)8^{4n-4}2^{4n-3}(1 - \alpha_N)^{(n-3)/64}.
\tag{4.55}
\]

Here \(5n - 1\) is a bound for the number of starting points of the paths, \(8^{4n-4}\) is a bound on the number of paths of \(4n - 3\) sites from a given point, \(2^{4n-3}\) a bound on the number of choices for the subset of the vertices which have to be black. Finally, \((1 - \alpha_N)^{(n-3)/64}\) is a bound for the probability that a given subset of \(n - 3\) vertices is black, because any such set contains a further subset of at least \((n-3)/5\) vertices, any two of which have distance of at least 2 between them, and have therefore independent colors. (4.54) for \( \ell = 1 \) and large \( N \) now follows from (4.42).

The remainder of the proof of (1.5) is now easy. The estimate (4.43) shows that under (1.2)

\[
P_p \{ (4.6) - (4.9) \text{ with } n \text{ replaced by } 56Nn \text{ hold} \} \geq 1 - C_{17}\exp(-C_{18}n).
\]

Also the \(\gamma\) of the hypotheses of Lemma 2 exist for all \(n\) if there exists an occupied path on \(\mathcal{M}\) from 0 to infinity in the sector \(\{(x, y) : x \geq 0, -x \leq y \leq 0\}\), and this event has a strictly positive probability (see Grimmett (1999), Theorem 11.55 and its proof). A similar argument applies to the existence of both \(\gamma\) and \(\delta\), as required in the hypotheses of Lemma 2. We conclude that there exists a constant \(C_{19} = C_{19}(p) > 0\) so that the probability that all hypotheses of Lemma 2 are fulfilled for \(n\) replaced by \(56Nn\) and \(k\) replaced by \(C_9n\) and for all \(n \geq n_0\) simultaneously, is at least

\[
C_{19} - \sum_{n \geq n_0} \left[ C_{10}e^{-C_{11}n} + C_{17}e^{-C_{18}n} \right].
\]

Fix \(n_0\) so large that the right hand side here is at least \(C_{19}/2\). Then by Lemmas 2 and 1, there is a probability of at least \(C_{19}/2\) that for all \(n \geq n_0\) all words of length \(C_5C_9(56)Nn^2\) are seen in \(S(56Nn)\). This proves (1.5).

References


