ON THE STRONG LAW OF LARGE NUMBERS FOR D-DIMENSIONAL ARRAYS OF RANDOM VARIABLES

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Abstract
In this paper, we provide a necessary and sufficient condition for general d-dimensional arrays of random variables to satisfy strong law of large numbers. Then, we apply the result to obtain some strong laws of large numbers for d-dimensional arrays of blockwise independent and blockwise orthogonal random variables.

1 Introduction
Let $\mathbb{Z}^d_+$, where $d$ is a positive integer, denote the positive integer $d$-dimensional lattice points. The notation $\mathbf{m} \prec \mathbf{n}$, where $\mathbf{m} = (m_1, m_2, \ldots, m_d)$ and $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d_+$, means that $m_i \leq n_i$, $1 \leq i \leq d$. Let $\{\alpha_i, 1 \leq i \leq d\}$ be positive constants. In Section 2, we provide a necessary and sufficient condition for

$$\lim_{|\mathbf{n}| \to \infty} \frac{S_{\mathbf{n}(\alpha)}}{|\mathbf{n}(\alpha)|} = 0 \text{ almost surely (a.s.)}$$

to hold. This condition springs from a recent result of Chobanyan, Levental and Mandrekar [1] which provided a condition for strong law of large numbers (SLLN) in the case $d = 1$ (see Chobanyan, Levental and Mandrekar [1, Theorem 3.3]). Some applications to SLLN for d-dimensional arrays of blockwise independent and blockwise orthogonal random variables are made in Section 3.

2 Result
We can now state our main result.
THEOREM 2.1. Let \( \{X_n, n \in \mathbb{Z}_+^d\} \) be a \( d \)-dimensional array of random variables and let \( \{\alpha_i, 1 \leq i \leq d\} \) be positive constants. For \( m = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \), set
\[
T_m = \frac{1}{|\mathbf{m}(\alpha)|} \max_{k \in I(m)} |\sum_{m < i < k} X_i|.
\]

Then
\[
\lim_{|m| \to \infty} T_m = 0 \quad \text{a.s.} \quad (2.1)
\]
if and only if
\[
\lim_{|n| \to \infty} \frac{S_n}{|\mathbf{n}(\alpha)|} = 0 \quad \text{a.s.} \quad (2.2)
\]

Proof. To prove Theorem 2.1, we will need the following lemma. The proof of the following lemma is just an application of Kronecker’s lemma with \( d \)-dimensional indices as was so kindly pointed out to the author by the referee.

LEMMA 2.1. Let \( \{x_n, n \in \mathbb{Z}_+^d\} \) be a \( d \)-dimensional array of constants, and let \( \{\alpha_i, 1 \leq i \leq d\} \) be a collection of positive constants. If
\[
\lim_{|n| \to \infty} x_n = 0, \quad (2.3)
\]
then
\[
\lim_{|n| \to \infty} \frac{1}{|\mathbf{n}(\alpha)|} \sum_{k < n} |\mathbf{k}(\alpha)| x_k = 0. \quad (2.4)
\]

Proof of Theorem 2.1. Let \( m = (m_1, \ldots, m_d), \ n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d \) with \( n \in I(m) \). Set
\[
\mathbf{n}^{(j)} = (n_1, \ldots, n_{j-1}, 2^{m_j-1} - 1, n_{j+1}, \ldots, n_d), \ 1 \leq j \leq d, \quad S_n^{(1)} = S_n^{(i)},
\]
\[
S_n^{(d)} = \sum_{i_1=2^{m_1-1}}^{n_1} \cdots \sum_{i_{d-1}=2^{m_{d-1}-1}}^{n_{d-1}} \sum_{i_d=1}^{2^{m_d-1}-1} X_{i_1, \ldots, i_d},
\]
and
\[
S_n^{(j)} = \sum_{i_1=2^{m_1-1}}^{n_1} \cdots \sum_{i_{j-1}=2^{m_{j-1}-1}}^{n_{j-1}} \sum_{i_j=1}^{2^{m_j-1}-1} \sum_{i_{j+1}=1}^{n_{j+1}} \sum_{i_d=1}^{2^{m_d-1}-1} X_{i_1, \ldots, i_d}, \ 2 \leq j \leq d - 1.
\]

Then
\[
S_n^{(j)} = S_n^{(j)} - \sum_{k=1}^{j-1} S_n^{(k)}, \ 2 \leq j \leq d. \quad (2.5)
\]

Assume that (2.1) holds. Since
\[
\frac{|S_n|}{|\mathbf{n}(\alpha)|} \leq \frac{1}{|\mathbf{m}(\alpha)|} \sum_{k < \mathbf{m}} |\mathbf{k}(\alpha)| T_k,
\]
the conclusion (2.2) holds by Lemma 2.1. Thus (2.1) implies (2.2). Now, assume that (2.2)
holds. Then
\[ \lim_{|m| \to \infty} \max_{n \in I(m)} \frac{S_n^{(1)}}{|n(\alpha)|} = 0 \text{ a.s.} \] (2.6)
For \(1 \leq j \leq d\), by (2.5), (2.6) and the induction method, we obtain
\[ \lim_{|m| \to \infty} \max_{n \in I(m)} \frac{S_n^{(j)}}{|n(\alpha)|} = 0 \text{ a.s.} \] (2.7)
Since
\[ S_n = \sum_{j=1}^{d} S_n^{(j)} + \sum_{i_1=2^{m_d-1}}^{n_1} \cdots \sum_{i_d=2^{m_d-1}}^{n_d} X_{(i_1, \ldots, i_d)}, \]
we have that
\[ |\sum_{i_1=2^{m_d-1}}^{n_1} \cdots \sum_{i_d=2^{m_d-1}}^{n_d} X_{(i_1, \ldots, i_d)}| \leq |S_n| + \sum_{j=1}^{d} |S_n^{(j)}|. \]
This implies
\[ T_m \leq 2^{\alpha_1 + \cdots + \alpha_d} \max_{n \in I(m)} \frac{|S_n| + \sum_{j=1}^{d} |S_n^{(j)}|}{|n(\alpha)|}. \] (2.8)
The conclusion (2.1) follows immediately from (2.2), (2.7) and (2.8).

3 Applications

In this section, we present some applications of Theorem 2.1. A \(d\)-dimensional array of random variables \(\{X_n, n \in \mathbb{Z}_d^d\}\) is said to be blockwise independent (resp., blockwise orthogonal) if for each \(k \in \mathbb{Z}_d^d\), the random variables \(\{X_i, i \in I(k)\}\) is independent (resp., orthogonal). The concept of blockwise independence for a sequence of random variables was introduced by Móricz [9]. Extensions of classical Kolmogorov SLLN (see, e.g., Chow and Teicher [2], p. 124) to the blockwise independence case were established by Móricz [9] and Gaposhkin [4]. Móricz [9] and Gaposhkin [4] also studied SLLN problem for sequence of blockwise orthogonal random variables.

Firstly, we establish a blockwise independence and \(d\)-dimensional version of the Kolmogorov SLLN.

THEOREM 3.1. Let \(\{X_n, n \in \mathbb{Z}_d^d\}\) be a \(d\)-dimensional array of mean 0 blockwise independent random variables and let \(\{\alpha_i, 1 \leq i \leq d\}\) be positive constants. If
\[ \sum_{n \in \mathbb{Z}_d^d} \frac{E|X_n|^p}{|n(\alpha)|^p} < \infty \text{ for some } 0 < p \leq 2, \] (3.1)
then SLLN
\[ \lim_{|n| \to \infty} \frac{S_n}{|n(\alpha)|} = 0 \text{ a.s.} \] (3.2)
obtains.

In the case \(0 < p \leq 1\), the independence hypothesis and the hypothesis that \(EX_n = 0, n \in \mathbb{Z}_d^d\) are superfluous.
Proof. We need the following lemma which was proved by Thanh [11] in the case $d = 2$. If $d$ is arbitrary positive integer, then the proof is similar and so is omitted.

**LEMMA 3.1.** Let $n \in \mathbb{Z}^d_+$ and let $\{X_i, i < n\}$ be a collection of $|n|$ mean 0 independent random variables. Then there exists a constant $C$ depending only on $p$ and $d$ such that

$$E(\max_{k < n} |S_k|^p) \leq C \sum_{i < n} E|X_i|^p \text{ for all } 0 < p \leq 2.$$ 

In the case $0 < p \leq 1$, the independence hypothesis and the hypothesis that $EX_i = 0, i < n$ are superfluous, and $C$ is given by $C = 1$. In the case $1 < p < 2$, $C$ is given by $C = 2(\frac{p}{p-1})^{pd}$. In the case $p = 2$, Lemma 3.1 was proved by Wichura [12] and $C$ is given by $C = 4d$.

**Proof of Theorem 3.1.** Define $T_m, m \in \mathbb{Z}^d_+$ as in Theorem 2.1. Note that for all $m \in \mathbb{Z}^d_+$,

$$E|T_m|^p = \frac{1}{|m(\alpha)|^p} E\left(\max_{k \in I(m)} |\sum_{m < i < k} X_i|\right)^p \leq \frac{C}{|m(\alpha)|^p} \sum_{i \in I(m)} E|X_i|^p \text{ (by Lemma 3.1)} \leq 2^{\alpha_1 + \cdots + \alpha_d} C \sum_{i \in I(m)} E|X_i|^p \frac{|i(\alpha)|^p}{|i(\alpha)|^p}.$$

It thus follows from (3.1) that $\sum_{m \in \mathbb{Z}^d_+} E|T_m|^p < \infty$ whence $\lim_{|m| \to \infty} T_m = 0$ a.s. The conclusion (3.2) follows immediately from Theorem 2.1.

The following theorem extends Theorem 3.1 and its part (ii) reduces to a result of Smythe [10] when the $\{X_n, n \in \mathbb{Z}^d_+\}$ are independent and $\alpha_1 = \cdots = \alpha_d = 1$.

**THEOREM 3.2.** Let $\{X_n, n \in \mathbb{Z}^d_+\}$ be a $d$-dimensional array of random variables and let $\{\alpha_i, 1 \leq i \leq d\}$ be positive constants. Assume that $\varphi(x)$ is a continuous functions on $[0, \infty)$, $\varphi(0) \geq 0, \varphi(x) > 0$ for $x > 0$, and

$$\sum_{n \in \mathbb{Z}^d_+} E(\varphi(|X_n|)) \varphi(|n(\alpha)|) < \infty. \quad (3.3)$$

If either

(i) $\varphi(x)/x \downarrow$, and $\varphi(x) \uparrow$

or

(ii) $\{X_n, n \in \mathbb{Z}^d_+\}$ are blockwise independent and have mean 0, and

$$\varphi(x)/x \uparrow, \; \varphi(x)/x^2 \downarrow,$$

then SLLN (3.2) obtains.

**Proof.** For $n \in \mathbb{Z}^d_+$, set

$$Y_n = X_n I(|X_n| \leq |n(\alpha)|),$$

$$Z_n = X_n I(|X_n| > |n(\alpha)|).$$
Consider the case (i) first. It follows from (3.3) that
\[
\sum_{n \in \mathbb{Z}^d} E\left|\frac{Y_n}{|\mathbf{n}(\alpha)|}\right| \leq \sum_{n \in \mathbb{Z}^d} E\left(\frac{\varphi(|Y_n|)}{\varphi(|\mathbf{n}(\alpha)|)}\right) \quad \text{(by the first condition of (i))}
\]
\[< \infty. \]

By Theorem 3.1,
\[
\lim_{|\mathbf{n}| \to \infty} \sum_{i < \mathbf{n}} Y_i \mathbf{n}(\alpha) = 0 \text{ a.s.} \quad (3.4)
\]

On the other hand
\[
\sum_{n \in \mathbb{Z}^d} P\{X_n \neq Y_n\} = \sum_{n \in \mathbb{Z}^d} P\{|X_n| > |\mathbf{n}(\alpha)|\}
\]
\[\leq \sum_{n \in \mathbb{Z}^d} P\{\varphi(|X_n|) \geq \varphi(|\mathbf{n}(\alpha)|)\}
\]
\[\leq \sum_{n \in \mathbb{Z}^d} E\left(\frac{\varphi(|X_n|)}{\varphi(|\mathbf{n}(\alpha)|)}\right)
\]
\[< \infty \quad \text{(by the second condition of (i))} \quad (3.5)
\]

By the Borel-Cantelli lemma,
\[
\lim_{|\mathbf{n}| \to \infty} \sum_{i < \mathbf{n}} (X_i - Y_i) \mathbf{n}(\alpha) = 0 \text{ a.s.} \quad (3.6)
\]

The conclusion (3.2) follows immediately from (3.4) and (3.5).

Now, consider the case (ii). It follows from (3.3) that
\[
\sum_{n \in \mathbb{Z}^d} E\left(\frac{Y_n - EY_n}{|\mathbf{n}(\alpha)|}\right)^2 \leq \sum_{n \in \mathbb{Z}^d} E\left(\frac{Y_n^2}{|\mathbf{n}(\alpha)|^2}\right)
\]
\[\leq \sum_{n \in \mathbb{Z}^d} E\left(\frac{\varphi(|Y_n|)}{\varphi(|\mathbf{n}(\alpha)|)}\right) \quad \text{(by the last condition of (ii))}
\]
\[< \infty \quad (3.6)
\]

and
\[
\sum_{n \in \mathbb{Z}^d} E\left|\frac{Z_n - EZ_n}{|\mathbf{n}(\alpha)|}\right| \leq 2 \sum_{n \in \mathbb{Z}^d} E\left|\frac{Z_n}{|\mathbf{n}(\alpha)|}\right|
\]
\[\leq 2 \sum_{n \in \mathbb{Z}^d} E\left(\frac{\varphi(|Z_n|)}{\varphi(|\mathbf{n}(\alpha)|)}\right) \quad \text{(by the second condition of (ii))}
\]
\[< \infty. \quad (3.7)
\]
By Theorem 3.1, the conclusion (3.6) implies

$$\lim_{|n| \to \infty} \frac{\sum_{i \prec n}(Y_i - EY_i)}{|n(\alpha)|} = 0 \text{ a.s.}$$  \hspace{1cm} (3.8)

and the conclusion (3.7) implies

$$\lim_{|n| \to \infty} \frac{\sum_{i \prec n}(Z_i - EZ_i)}{|n(\alpha)|} = 0 \text{ a.s.}$$  \hspace{1cm} (3.9)

The conclusion (3.2) follows immediately from (3.8) and (3.9).

REMARK 3.1. (i) According to the discussion in Smythe [10], the proof of part (ii) of Theorem 3.2 was based on the “Khintchin-Kolmogorov convergence theorem, Kronecker lemma approach”. But it seems that the Kronecker lemma for $d$-dimensional arrays when $d \geq 2$ is not such a good tool as in the study of the SLLN for the case $d = 1$ (see Mikosch and Norvaisa [6]). Moreover, in the blockwise independence case, according to an example of Móricz [9], the conclusion of Theorem 3.1 (or part (ii) of Theorem 3.2) cannot in general be reached through the well-know Kronecker lemma approach for proving SLLNs even when $d = 1$.

(ii) Chung [3] proved part (i) of Theorem (3.2) (for the case $d = 1$ only) by the Kolmogorov three series theorem and the Kronecker lemma. So in his proof, the independence assumption must be required.

We now establish the Marcinkiewicz-Zygmund SLLN for $d$-dimensional arrays of blockwise independent identically distributed random variables. The following theorem reduces to a result of Gut [5] when the $\{X_n, n \in \mathbb{Z}_d^d\}$ are independent.

THEOREM 3.3. Let $\{X, X_n, n \in \mathbb{Z}_d^d\}$ be a $d$-dimensional array of blockwise independent identically distributed random variables with $EX = 0$, $E(|X|^r (\log^+ |X|)^{d-1}) < \infty$ for some $1 \leq r < 2$. Then SLLN

$$\lim_{|n| \to \infty} \frac{S_n}{|n|^{1/r}} = 0 \text{ a.s.}$$  \hspace{1cm} (3.10)

obtains.

Proof. According to the proof of Lemma 2.2 of Gut [5],

$$\sum_{n \in \mathbb{Z}_d^d} \frac{E(Y_n - EY_n)^2}{|n|^{2/r}} < \infty$$  \hspace{1cm} (3.11)

where $Y_n = X_n(|X_n| \leq |n|^{1/r})$, $n \in \mathbb{Z}_d^d$. And similarly, we also have

$$\sum_{n \in \mathbb{Z}_d^d} \frac{E|Z_n - EZ_n|}{|n|^{1/r}} < \infty$$  \hspace{1cm} (3.12)

where $Z_n = X_n(|X_n| > |n|^{1/r})$, $n \in \mathbb{Z}_d^d$. By Theorem 3.1 (with $\alpha_i = 1/r, 1 \leq i \leq d$), the conclusion (3.10) follows immediately from (3.11) and (3.12).

Finally, we establish the SLLN for $d$-dimensional arrays of blockwise orthogonal random variables. The following theorem is a blockwise orthogonality version of Theorem 1 of Móricz [8]
and its proof is based on the $d$-dimensional version of the Rademacher-Mensov inequality (see Móricz [7]) and the method used in the proof of Theorem 3.1.

THEOREM 3.4. Let $\{X_n, n \in \mathbb{Z}_+^d\}$ be a $d$-dimensional array of blockwise orthogonal random variables and let $\{\alpha_i, 1 \leq i \leq d\}$ be positive constants. If

$$\sum_{n \in \mathbb{Z}_+^d} E\left|X_n\right|^2 \prod_{i=1}^d \left[\log(n_i + 1)\right]^2 < \infty,$$

then SLLN (3.2) obtains.

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**References**


