Lp ESTIMATES FOR SPDE WITH DISCONTINUOUS COEFFICIENTS IN DOMAINS

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Abstract. Stochastic partial differential equations of divergence form with discontinuous and unbounded coefficients are considered in $C^1$ domains. Existence and uniqueness results are given in weighted $L_p$ spaces, and Hölder type estimates are presented.

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1. Introduction

Let $G$ be an open set in $\mathbb{R}^d$. We consider parabolic stochastic partial differential equations of the form

$$du = (D_i(a^{ij}u_{xj} + b^iu + f^i) + \bar{b}^iu_{x_i} + cu + \bar{f}) dt + (\nu^ku + g^k) dw^k_t, \quad (1.1)$$

given for $x \in G, t \geq 0$. Here $w^k_t$ are independent one-dimensional Wiener processes, $i$ and $j$ go from 1 to $d$, and $k$ runs through $\{1, 2, \ldots\}$. The coefficients $a^{ij}, b^i, \bar{b}^i, c, \nu^k$ and the free terms $f^i, \bar{f}, g^k$ are random functions depending on $t > 0$ and $x \in G$.

This article is a natural continuation of the article [15], where $L_p$ estimates for the equation

$$du = D_i(a^{ij}u_{xj} + f^i) dt + (\nu^ku + g^k) dw^k_t \quad (1.2)$$

with discontinuous coefficients was constructed on $\mathbb{R}^d$.

Our approach is based on Sobolev spaces with or without weights, and we present the unique solvability result of equation (1.1) on $\mathbb{R}^d, \mathbb{R}^d_+$ (half space) and on bounded $C^1$ domains. We show that $L_p$-norm of $u_x$ can be controlled by $L_p$-norms of $f^i, \bar{f}$ and $g$ if $p$ is sufficiently close to 2.

Pulvirenti [13] showed by example that without the continuity of $a^{ij}$ in $x$ one can not fix $p$ even for deterministic parabolic equations. For an $L_p$ theory of linear SPDEs with continuous coefficients on domains, we refer to [1], [2] and [7].

Actually $L_2$ theory for type (1.1) with bounded coefficients was developed long times ago on the basis of monotonicity method, and an account of it can be found in [14]. But our results are new even for $p = 2$ (and probably even for deterministic equation) since, for instance, we are only assuming the functions

$$\rho b^i, \quad \rho \bar{b}^i, \quad \rho^2 c, \quad \rho \nu^k$$

are bounded, where $\rho(x) = \text{dist}(x, \partial G)$. Thus we are allowing our coefficients to blow up near the boundary of $G$.

An advantage of $L_p(p > 2)$ theory can be found, for instance, in [16], where solvability of some nonlinear SPDEs was presented with the help of $L_p$ estimates for linear SPDEs with discontinuous coefficients. Also we will see that some Hölder type estimates are valid only for $p > 2$ (Corollary 2.5).

We finish the introduction with some notations. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, \ldots, x^d)$, $\mathbb{R}^d_+ = \{ x \in \mathbb{R}^d : x^1 > 0 \}$ and $B_r(x) := \{ y \in \mathbb{R}^d : |x - y| < r \}$. For $i = 1, \ldots, d$, multi-indices
\[ \alpha = (\alpha_1, \ldots, \alpha_d), \alpha_i \in \{0,1,2,\ldots\}, \text{ and functions } u(x) \text{ we set} \]

\[ u_{x^i} = \partial u / \partial x^i = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdots D_d^{\alpha_d} u, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d. \]

2. Main Results

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space, \(\{\mathcal{F}_t, t \geq 0\}\) be an increasing filtration of \(\sigma\)-fields \(\mathcal{F}_t \subset \mathcal{F}\), each of which contains all \((\mathcal{F}, P)\)-null sets. By \(\mathcal{P}\) we denote the predictable \(\sigma\)-field generated by \(\{\mathcal{F}_t, t \geq 0\}\) and we assume that on \(\Omega\) we are given independent one-dimensional Wiener processes \(w_t^1, w_t^2, \ldots\), each of which is a Wiener process relative to \(\{\mathcal{F}_t, t \geq 0\}\).

Fix an increasing function \(\kappa_0\) defined on \([0, \infty)\) such that \(\kappa_0(\varepsilon) \to 0\) as \(\varepsilon \downarrow 0\).

Assumption 2.1. The domain \(G \subset \mathbb{R}^d\) is of class \(C^1\). In other words, there exist constants \(r_0, K_0 > 0\) such that for any \(x_0 \in \partial G\) there exists a one-to-one continuously differentiable mapping \(\Psi\) from \(B_{r_0}(x_0)\) onto a domain \(J \subset \mathbb{R}^d\) such that

(i) \(J := \Psi(B_{r_0}(x_0) \cap G) \subset \mathbb{R}^d_+\) and \(\Psi(x_0) = 0\);
(ii) \(\Psi(B_{r_0}(x_0) \cap \partial G) = J \cap \{y \in \mathbb{R}^d : y^+ = 0\}\);
(iii) \(|\Psi'|_{C^1(B_{r_0}(x_0))} \leq K_0\) and \(|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \leq K_0|y_1 - y_2|\) for any \(y_1, y_2 \in J\);
(iv) \(|\Psi_x(x_1) - \Psi_x(x_2)| \leq \kappa_0(|x_1 - x_2|)\) for any \(x_1, x_2 \in B_{r_0}(x_0)\).

Assumption 2.2. (i) For each \(x \in G\), the functions \(a^{ij}(t, x), b^i(t, x), b^j(t, x), c(t, x)\) and \(\nu^k(t, x)\) are predictable functions of \((\omega, t)\).

(ii) There exist constants \(\lambda, \Lambda \in (0, \infty)\) such that for any \(\omega, t, x\) and \(\xi \in \mathbb{R}^d\),

\[ \lambda|\xi|^2 \leq a^{ij}\xi^i\xi^j \leq \Lambda|\xi|^2. \]

(iii) For any \(x, t\) and \(\omega\),

\[ \rho(x)|b^i(t, x)| + \rho(x)|\bar{b}^i(t, x)| + \rho(x)^2|c(t, x)| + \rho(x)|\nu^k(t, x)| \leq K. \]

(iv) There is control on the behavior of \(b^i, \bar{b}^i, c, \nu\) near \(\partial G\), namely,

\[ \lim_{\rho(x) \to 0} \sup_{t, \omega} \rho(x)(|b^i(t, x)| + |\bar{b}^i(t, x)|) = 0. \quad (2.1) \]

To describe the assumptions of \(f^i, \bar{f}^i\) and \(g\) we use Sobolev spaces introduced in [7], [8] and [12]. If \(n\) is a non negative integer, then

\[ H^n_p = H^n_p(\mathbb{R}^d) = \{u : u, Du, \ldots, D^\alpha u \in L_p : |\alpha| \leq n\}, \]

\[ L_{p, \theta}(G) := H^0_{p, \theta}(G) = L_p(G, \rho^{-d}dx), \quad \rho(x) := \text{dist}(x, \partial G), \]

\[ H^n_{p, \theta}(G) := \{u : u, \rho u_x, \ldots, \rho^{|\alpha|} D^\alpha u \in L_{p, \theta}(G) : |\alpha| \leq n\}. \quad (2.2) \]
In general, by $H^\gamma_p = H^\gamma_p(\mathbb{R}^d) = (1 - \Delta)^{-\gamma/2}L_p$ we denote the space of Bessel potential, where

$$\|u\|_{H^\gamma_p} = \|(1 - \Delta)^{\gamma/2}u\|_{L_p},$$

and the weighted Sobolev space $H^\gamma_{p,\theta}(G)$ is defined as the set of all distributions $u$ on $G$ such that

$$\|u\|^p_{H^\gamma_{p,\theta}(G)} := \sum_{n=-\infty}^{\infty} e^{\theta n} \|\zeta_n(e^n \cdot)u(e^n \cdot)\|^p_{H^\gamma_p} < \infty, \quad (2.3)$$

where $\{\zeta_n : n \in \mathbb{Z}\}$ is a sequence of functions $\zeta_n \in C_0^\infty(G)$ such that

$$\sum_{n} \zeta_n \geq c > 0, \quad \forall x \in \mathbb{R},$$

and define $\zeta_n(x) = \zeta(e^n x)$, then (2.3) becomes

$$\|u\|^p_{H^\gamma_{p,\theta}} := \sum_{n=-\infty}^{\infty} e^{\theta n} \|\zeta(\cdot)u(e^n \cdot)\|^p_{H^\gamma_p} < \infty. \quad (2.5)$$

It is known that up to equivalent norms the space $H^\gamma_{p,\theta}$ is independent of the choice $\zeta$, and $H^\gamma_{p,\theta}(G)$ and its norm are independent of $\{\zeta_n\}$ if $G$ is bounded.

We use above notations for $\ell_2$-valued functions $g = (g_1, g_2, \ldots)$. For instance

$$\|g\|_{H^\gamma_{p}(\ell_2)} = \|(1 - \Delta)^{\gamma/2}g\|_{\ell_2} \cdot L_p.$$

For any stopping time $\tau$, denote $\{0, \tau\} = \{\omega, t \in (0, \tau) \} \cup \{\omega, t = \tau(\omega)\}$, where

$$\mathcal{H}_p(\tau) = L_p((0, \tau], \mathcal{P}, H^\gamma_p), \quad \mathcal{H}_{p,\theta}(G, \tau) = L_p((0, \tau], \mathcal{P}, H^\gamma_{p,\theta}(G)),$$

and

$$\mathbb{H}_{p,\theta}(\tau) = L_p((0, \tau], \mathcal{P}, H^\gamma_{p,\theta}), \quad \mathbb{H}_\cdot(\tau) = H^\gamma_{p,\theta}(\cdot).$$

Fix (see [5]) a bounded real-valued function $\psi$ defined in $\bar{G}$ such that for any multi-index $\alpha,$

$$[\psi]_{\alpha} := \sup_{G} \rho_{\alpha}(x)|D^{\alpha}\psi(x)| < \infty$$

and the functions $\psi$ and $\rho$ are comparable in a neighborhood of $\partial G$. As in [11], by $M^\alpha$ we denote the operator of multiplying by $(x^1)^{\alpha}$ and $M = M^1$. Define

$$U^\gamma_p = L_p(\Omega, \mathcal{F}_0, H^\gamma_p^{-2/p}), \quad U^\gamma_{p,\theta} = M^{1-2/p}L_p(\Omega, \mathcal{F}_0, H^\gamma_{p,\theta}^{-2/p}),$$

and

$$U^\gamma_{p,\theta}(G) = \psi^{1-2/p}L_p(\Omega, \mathcal{F}_0, H^\gamma_{p,\theta}^{-2/p}(G)).$$
By $\mathcal{S}_{p,\theta}^\gamma(G, \tau)$ we denote the space of all functions $u \in \psi H_{p,\theta}^\gamma(G, \tau)$ such that $u(0, \cdot) \in U_{p,\theta}^\gamma(G)$ and for some $f \in \psi^{-1}H_{p,\theta}^{\gamma-2}(G, \tau)$, $g \in H_{p,\theta}^{\gamma-1}(G, \tau)$,

$$du = f \, dt + g^k \, dw^k_t,$$

in the sense of distributions. In other words, for any $\phi \in C_0^\infty(G)$, the equality

$$(u(t, \cdot), \phi) = (u(0, \cdot), \phi) + \int_0^t (f(s, \cdot), \phi) \, ds + \sum_{0}^{\infty} \int_0^t (g^k(s, \cdot), \phi) \, dw^k_s$$

holds for all $t \leq \tau$ with probability 1.

The norm in $\mathcal{S}_{p,\theta}^\gamma(G, \tau)$ is introduced by

$$\|u\|_{\mathcal{S}_{p,\theta}^\gamma(G, \tau)} = \|\psi^{-1}u\|_{H_{p,\theta}^\gamma(G, \tau)} + \|\psi f\|_{H_{p,\theta}^{\gamma-2}(G, \tau)}$$

$$+ \|g\|_{H_{p,\theta}^{\gamma-1}(G, \tau)} + \|u(0, \cdot)\|_{U_{p,\theta}(G)}.$$  

It is easy to check that up to equivalent norms the space $\mathcal{S}_{p,\theta}^\gamma(G, \tau)$ and its norm are independent of the choice of $\psi$ if $G$ is bounded.

We write $u \in \mathcal{S}_{p,\theta}^\gamma(\tau)$ if $u \in M H_{p,\theta}^\gamma(\tau)$ satisfies (2.6) for some $f \in M^{-1}H_{p,\theta}^{\gamma-2}(\tau)$, $g \in H_{p,\theta}^{\gamma-1}(\tau, \ell_2)$, and we define

$$\|u\|_{\mathcal{S}_{p,\theta}^\gamma(\tau)} = \|M^{-1}u\|_{M H_{p,\theta}^\gamma(\tau)} + \|M f\|_{H_{p,\theta}^{\gamma-2}(\tau)}$$

$$+ \|g\|_{H_{p,\theta}^{\gamma-1}(\tau)} + \|u(0, \cdot)\|_{U_{p,\theta}(G)}.$$  

Similarly we define stochastic Banach space $\mathcal{H}_{p}^\gamma(\tau)$ on $\mathbb{R}^d$ (and its norm) by formally taking $\psi = 1$ and replacing $H_{p,\theta}^\gamma(G), U_{p,\theta}(G)$ by $H_{p}^\gamma, U_{p}^\gamma$ respectively, in the definition of the space $\mathcal{S}_{p,\theta}^\gamma(G, \tau)$.

We drop $\tau$ in the notations of appropriate Banach spaces if $\tau \equiv \infty$. Note that if $G = \mathbb{R}^d$, then $\mathcal{S}_{p,\theta}^\gamma(G, \tau)$ is slightly different from $\mathcal{S}_{p,\theta}^\gamma(\tau)$ since $\psi(x)$ is bounded. Finally we define

$$\mathcal{S}_{p,\theta}^\gamma(\ldots) = \mathcal{S}_{p,\theta}^\gamma(\ldots) \cap \{u : u(0, \cdot) = 0\};$$

$$\mathcal{H}_{p,\theta}^\gamma(\ldots) = \mathcal{H}_{p,\theta}^\gamma(\ldots) \cap \{u : u(0, \cdot) = 0\}.$$  

Some properties of the spaces $H_{p,\theta}^\gamma, \mathcal{S}_{p,\theta}^\gamma(G, \tau)$ and $\mathcal{H}_{p}^\gamma(\tau)$ are collected in the following lemma (see [3],[7], [8] and [12] for detail). From now on we assume that

$$p \geq 2, \quad d - 1 < \theta < d - 1 + p.$$

**Lemma 2.3.** (i) The following are equivalent:

(a) $u \in H_{p,\theta}^\gamma(G)$,

(b) $u \in H_{p,\theta}^{\gamma-1}(G)$ and $\psi Du \in H_{p,\theta}^{\gamma-1}(G)$,

(c) $u \in H_{p,\theta}^{\gamma-1}(G)$ and $D(\psi u) \in H_{p,\theta}^{\gamma-1}(G)$. 

In addition, under either of these three conditions
\[
\|u\|_{H^{\gamma}_{p,\theta}(G)} \leq N \|\nu u_x\|_{H^{\gamma-1}_{p,\theta}(G)} \leq N \|u\|_{H^{\gamma}_{p,\theta}(G)}, \tag{2.7}
\]
\[
\|u\|_{H^{\gamma}_{p,\theta}(G)} \leq N \|\nu u\|_{H^{\gamma-1}_{p,\theta}(G)} \leq N \|u\|_{H^{\gamma}_{p,\theta}(G)}, \tag{2.8}
\]

(ii) For any \(\nu, \gamma \in \mathbb{R}\), \(\nu u H^{\gamma}_{p,\theta}(G) = H^{\gamma}_{p,\theta-\nu}(G)\), and
\[
\|u\|_{H^{\gamma}_{p,\theta-\nu}(G)} \leq N \|\nu u\|_{H^{\gamma}_{p,\theta}(G)} \leq N \|u\|_{H^{\gamma}_{p,\theta-\nu}(G)}.
\]

(iii) There exists a constant \(N\) depending only on \(d, p, \gamma, T\) (and \(\theta\)) such that for any \(t \leq T\),
\[
\|u\|_{H^{\gamma}_{p,\theta}(G,t)} \leq N \int_0^t \|u\|^p_{S^{\gamma+1}_{p,\theta}(G,s)} \, ds \leq N t \|u\|^p_{S^{\gamma+1}_{p,\theta}(G,t)}, \tag{2.9}
\]
\[
\|u\|^p_{H^{\gamma}_{p}(t)} \leq N \int_0^t \|u\|^p_{H^{\gamma+1}_{p}(s)} \, ds \leq N t \|u\|^p_{H^{\gamma+1}_{p}(t)}. \tag{2.10}
\]

(iv) Let \(\gamma - d/p = m + \nu\) for some \(m = 0, 1, \ldots\) and \(\nu \in (0, 1)\), then for any \(k \leq m\),
\[
|\nu^{k+\theta/p}D^k u|_{C^0} + [\nu^{m+\nu+\theta/p}D^m u]_{C^0} \leq N \|u\|_{H^{\gamma}_{p,\theta}(G)}.
\]

(v) Let
\[
2/p < \alpha < \beta < 1.
\]
Then for any \(u \in S^\gamma_{p,\theta,0}(G, \tau)\) and \(0 \leq s < t \leq \tau\),
\[
E|\nu^{\beta-1} (u(t) - u(s))|^p_{H^{\gamma-\beta}_{p,\theta}(G)} \leq N |t - s|^{p\beta/2 - 1} \|u\|^p_{S^\gamma_{p,\theta}(G, \tau)}, \tag{2.11}
\]
\[
E|\psi^{\beta-1} u|^p_{C^{0,\alpha/2 - 1/p}(\{0, \tau\}, H^{\gamma-\beta}_{p,\theta}(G))} \leq N \|u\|^p_{S^\gamma_{p,\theta}(G, \tau)}. \tag{2.12}
\]

Here are our main results.

**Theorem 2.4.** Assume \(G\) is bounded and \(\tau \leq T\). Under the above assumptions, there exist \(p_0 = p_0(\lambda, \Lambda, d) > 2\) and \(\chi = \chi(p, d, \lambda, \Lambda) > 0\) such that if \(p \in [2, p_0)\) and \(\theta \in (d - \chi, d + \chi)\), then

(i) for any \(f^i \in L_{p,\theta}(G, \tau)\), \(\bar{f} \in \psi^{-1}H^{-1}_{p,\theta}(G, \tau)\), \(g \in L_{p,\theta}(G, \tau)\) and \(u_0 \in U^1_{p,\theta}(G)\) equation (1.1) admits a unique solution \(u \in S^1_{p,\theta}(G, \tau)\),

(ii) for this solution
\[
\|\psi^{-1} u\|_{H^1_{p,\theta}(G, \tau)} \leq N (\|f^i\|_{L_{p,\theta}(G, \tau)} + \|\bar{f}\|_{H^{-1}_{p,\theta}(G, \tau)} + \|g\|_{L_{p,\theta}(G, \tau)} + \|u_0\|_{U^1_{p,\theta}(G)}), \tag{2.13}
\]
where the constant \(N\) is independent of \(f^i, \bar{f}, g, u\) and \(u_0\).

Lemma 2.3 (iv) and (v) yield the following results. It is crucial that \(p\) is bigger than 2.
Corollary 2.5. Let \( u \in \mathfrak{H}_{p,0,0}^1(G, \tau) \) be the solution of (1.1) and
\[
\frac{2}{p} < \alpha < \beta \leq 1.
\]
(i) Then for any \( 0 \leq s < t \leq \tau \),
\[
E[|\psi^{\beta-1}(u(t) - u(s))|^{1-\beta}_{\mathcal{H}_{p,0}^1(G)}] \leq N|t - s|^{p-2-1}C(f, \tilde{f}, g, \theta)
\]
(2.14)
\[
E[|\psi^{\beta-1}u|^{p}_{C^{\alpha/2-1/r}([0,\tau];\mathcal{H}_{p,0}^1(G))}] \leq NC(f, \tilde{f}, g, \theta),
\]
(2.15)
where \( C(f, \tilde{f}, g, \theta) := \|f\|_{L_{p,0}^1(G, \tau)} + \|\psi\tilde{f}\|_{\mathcal{H}_{p,0}^{1-1}(G, \tau)} + \|g\|_{L_{p,0}^1(G, \tau)} \).
(ii) If \( d \leq 2, 1 - d/p =: \nu \), then
\[
E\int_0^\tau (|\psi^{\theta-1}u|^{p}_{C^\nu} + |\psi^{\theta-4}u|^{p}_{C^\nu(G)}) dt \leq NC(f, \tilde{f}, g, \theta),
\]
(2.16)
thus if \( \theta \leq d \), then the function \( u \) itself is Hölder continuous in \( x \).

The following corollary shows that if some extra conditions are assumed, then the solutions are Hölder continuous in \( (t, x) \) (regardless of the dimension \( d \)).

Corollary 2.6. Let \( u \in \mathfrak{H}_{p,d,0}^1(G, T) \) be the solution of (1.1). Assume that \( b^i, \tilde{b}, c \) are bounded, \( \nu = 0 \) and
\[
1 - 2/q - d/r > 0, \quad q \geq r > 2,
\]
\[
f^i, f, g \in L_q(\Omega \times [0, T], \mathcal{P}, L_r(G)).
\]
Then there exists \( \alpha = \alpha(q, r, d, G) > 0 \) such that
\[
E[u]^\alpha_{C^\alpha([0,T])} < \infty.
\]
(2.17)
Proof. It is shown in [3] that under the conditions of the corollary, there is a solution \( v \in \mathfrak{H}_{2,d,0}^1(G, T) \) satisfying (2.17). By the uniqueness result (Theorem 2.4) in the space \( \mathfrak{H}_{2,d}^1(G, T) \), we conclude that \( u = v \) and thus \( v \in \mathfrak{H}_{p,d}^1(G, T) \).

We will see that the proof of Theorems 2.4 depends also on the following results on \( \mathbb{R}^d_+ \) and \( \mathbb{R}^d \).

Theorem 2.7. Assume that
\[
x^1|b^i(t, x)| + x^1|\tilde{b}^i(t, x)| + (x^1)^2|c(t, x)| + x^1|\nu(t, x)| \leq \beta, \quad \forall \omega, t, x.
\]
Then there exist \( p_0 = p_0(\lambda, \Delta, d) > 2 \), \( \beta_0 = \beta_0(p, d, \lambda, \Lambda) \in (0, 1) \) and \( \chi = \chi(p, d, \lambda, \Lambda) > 0 \) such that if
\[
\beta \leq \beta_0, \quad p \in [2, p_0), \quad d - \chi < \theta < d + \chi,
\]
(2.18)
then for any \( f^i \in \mathbb{L}_{p, \theta}(\tau), \tilde{f} \in M^{-1}_{p, \theta}(\tau), g \in \mathbb{L}_{p, \theta} \) and \( u_0 \in U^1_{p, \theta} \) equation (1.1) with initial data \( u_0 \) admits a unique solution \( u \) in the class \( \mathcal{H}_{p, \theta}^1(\tau) \) and for this solution,

\[
\|M^{-1}u\|_{\mathcal{H}_{p, \theta}^1(\tau)} \leq N(\|f^i\|_{\mathcal{L}_{p, \theta}(\tau)} + \|\tilde{f}\|_{H^{-1}_{p, \theta}(\tau)} + \|g\|_{\mathcal{L}_{p, \theta}(\tau)} + \|u_0\|_{U^1_{p, \theta}}), \tag{2.19}
\]

where \( N \) depends only \( d, p, \theta, \lambda \) and \( \Lambda \).

**Theorem 2.8.** Assume that

\[
|b^i(t, x)| + |\tilde{b}^i(t, x)| + |c(t, x)| + |\nu(t, x)| \leq K, \quad \forall \omega, t, x.
\]

Then there exists \( p_0 > 2 \) such that if \( p \leq [2, p_0) \), then for any \( f^i \in \mathbb{L}_{p}(\tau), \tilde{f} \in H^{-1}_{p}(\tau), g \in \mathbb{L}_{p}(\tau), u_0 \in U^1_{p} \) equation (1.1) with initial data \( u_0 \) admits a unique solution \( u \) in the class \( \mathcal{H}_{p}^1(\tau) \) and for this solution,

\[
\|u\|_{\mathcal{H}_{p}^1(\tau)} \leq N(\|f^i\|_{\mathcal{L}_{p}(\tau)} + \|\tilde{f}\|_{H^{-1}_{p}(\tau)} + \|g\|_{\mathcal{L}_{p}(\tau)} + \|u_0\|_{U^1_{p}}), \tag{2.20}
\]

where \( N \) depends only \( d, p, \lambda, \Lambda, K \) and \( T \).

### 3. Proof of Theorem 2.7

First we prove the following lemmas.

**Lemma 3.1.** Let \( f = (f^1, f^2, \ldots, f^d), g = (g^1, g^2, \ldots) \in \mathbb{L}_{2,d}(T) \) and \( u \in \mathcal{H}_{2,d,0}^1(T) \) be a solution of

\[
du = (\Delta u + f^i_{x^i})dt + g^k dw^k_t. \tag{3.1}
\]

Then

\[
\|u_x\|^2_{L^2_{2,d}(T)} \leq \|f\|^2_{L^2_{2,d}(T)} + \|g\|^2_{L^2_{2,d}(T)}. \tag{3.2}
\]

**Proof.** It is well known (see [11]) that (3.1) has a unique solution \( u \in \mathcal{H}_{p, d, 0}^1(T) \) and

\[
\|u_x\|^p_{L^p_{2,d}(T)} \leq N(p)(\|f\|^p_{L^p_{2,d}(T)} + \|g\|^p_{L^p_{2,d}(T)}). \tag{3.3}
\]

We will show that one can take \( N(2) = 1 \). Let \( \Theta \) be the collections of the form

\[
f(t, x) = \sum_{i=1}^{m} I_{(\tau_i, 1, \tau_{i+1}]}(t)f_i(x),
\]

where \( f_i \in C^\infty_0(\mathbb{R}^d) \) and \( \tau_i \) are stopping times, \( \tau_i \leq \tau_{i+1} \leq T \). It is well known that the set \( \Theta \) is dense in \( H_{p, \theta}(T) \) for any \( \gamma, \theta \in \mathbb{R} \). Also the collection of sequences \( g = (g^k) \), such that each \( g_k \in \Theta \) and only finitely many of \( g_k \) are different from zero, is dense in \( H_{p, \theta}(T, \ell_2) \). Thus by considering approximation argument, we may assume that \( f \) and \( g \) are of this type.
We continue \( f(t,x) \) to be an even function and \( g(t,x) \) to be an odd function of \( x^1 \). Then obviously \( f, g \in H_p^\gamma(T) \) for any \( \gamma \) and \( p \). By Theorem 5.1 in [7], equation (3.1) considered in the whole \( \mathbb{R}^d \) has a unique solution \( v \in \mathcal{H}_p^1 \) and \( v \in \mathcal{H}_p^\gamma \) for any \( \gamma \). Also by the uniqueness it follows that \( v \) is an odd function of \( x^1 \) and vanishes at \( x^1 = 0 \). Moreover remembering the fact that \( v \) satisfies
\[
dv = \Delta v \, dt
\]
outside the support of \( f \) and \( g \), we conclude (see the proof of Lemma 4.2 in [10] for detail) that \( v \in \mathcal{H}_p^1 \) for any \( \gamma \).

Thus, both \( u \) and \( v \) satisfy (3.1) considered in \( \mathbb{R}^d_+ \) and belong to \( \mathcal{H}_p^1 \). By the uniqueness result (Theorem 3.3 in [11]) on \( \mathbb{R}^d_+ \), we conclude that \( u = v \).

Finally, we see that (3.2) follows from Itô’s formula. Indeed (remember that \( u \) is infinitely differentiable and vanishes at \( x^1 = 0 \)),
\[
|u(t,x)|^2 = \int_0^t (2u \Delta u + 2uf_{x^i} + |g|_{\ell_2}^2) \, dt + 2 \int_0^t ug \, dw^k,
\]
therefore
\[
0 \leq E \int_{\mathbb{R}^d_+} |u(t,x)|^2 \, dx = -2E \int_0^t \int_{\mathbb{R}^d_+} |Du(s,x)|^2 \, dxdt -2E \int_0^t \int_{\mathbb{R}^d_+} f^i D^i u \, dxdt + E \int_0^t \int_{\mathbb{R}^d_+} |g|_{\ell_2}^2 \, dxdt
\]
\[
\leq -E \int_0^t \int_{\mathbb{R}^d_+} |Du(s,x)|^2 \, dxdt + E \int_0^t \int_{\mathbb{R}^d_+} |f|^2 \, dxdt + E \int_0^t \int_{\mathbb{R}^d_+} |g|_{\ell_2}^2 \, dxdt.
\]

\[ \square \]

Lemma 3.2. There exists \( p_0 = p_0(\lambda, \Lambda, d) > 2 \) such that if \( p \in [2, p_0) \) and \( u \in \mathcal{H}_p^{1,0}(T) \) is a solution of
\[
du = D_i(a^{ij}u_{x^j} + f^i)dt + g^k dw^k_t, \tag{3.4}
\]
then
\[
\|u_x\|_{L_p(T)} \leq N\left(\|f\|_{L_p(T)} + \|g\|_{L_p(T)}\right), \tag{3.5}
\]
where \( N \) is independent of \( T \).
**Proof.** We repeat arguments in [15]. Take $N(p)$ from (3.3). By (real-
valued version) Riesz-Thorin theorem we may assume that $N(p) \searrow 1$
as $p \searrow 2$. Indeed, consider the operator

$$
\Phi : (f^i, g) \rightarrow Du,
$$

where $u \in \mathcal{H}_{p,d,0}^1$ is the solution of (3.1). Then for any $r > 2$ and

$$
p \in [2, r],
$$

$$
\|\Phi\|_p \leq (1 - \alpha) \|\Phi\|_r^{(1/2-1/p)/(1/2-1/r)} \rightarrow 1.
$$

Denote $A := (a^{ij}), \kappa := \frac{\lambda + \Lambda}{2}$ and observe that the eigenvalues of

$$
(A - \kappa I)
$$

satisfy

$$
-(\Lambda - \lambda)/2 = \lambda - \kappa \leq \lambda_1 - \kappa \leq \cdots \leq \lambda_d - \kappa = (\Lambda - \lambda)/2,
$$

and therefore for any $\xi \in \mathbb{R}^d$,

$$
|((a^{ij} - \kappa I)\xi) | \leq \frac{\Lambda - \lambda}{2} |\xi|.
$$

(3.6)

Assume that $v \in \mathcal{H}_{p,d,0}^1(T)$ satisfies

$$
dv = (\kappa \Delta v + f^i_{x^i}) dt + g^k dw^k_t.
$$

Then $\bar{v}(t, x) := v(t, \sqrt{k}x)$ satisfies

$$
d\bar{v} = (\Delta \bar{v} + \bar{f}^i_{x^i}) dt + \bar{g}^k dw^k_t,
$$

where $\bar{f}^i(t, x) = \frac{1}{\sqrt{k}} f^i(t, \sqrt{k}x)$ and $\bar{g}^k(t, x) = g^k(t, \sqrt{k}x)$. Thus by

(3.3),

$$
\|u_x\|_{L^p_{p,d}(T)}^{p} \leq \frac{N(p)}{\kappa^p} \|f\|_{L^p_{p,d}(T)}^{p} + \frac{N(p)}{\kappa^{p/2}} \|g\|_{L^p_{p,d}(T)}^{p}.
$$

(3.7)

Therefore we conclude that if $u \in \mathcal{H}_{p,d,0}^1(T)$ is a solution of (3.4), then $u$
satisfies

$$
du = (\kappa \Delta u + (f^i + (A - \kappa I)u_{x^i})_{x^i}) dt + g^k dw^k_t,
$$

and

$$
\|u_x\|_{L^p_{p,d}(T)}^{p} \leq \frac{N(p)}{\kappa^p} \|F\|_{L^p_{p,d}(T)}^{p} + \frac{N(p)}{\kappa^{p/2}} \|g\|_{L^p_{p,d}(T)}^{p},
$$

where $F^i = (A - \kappa I)u_{x^i} + f^i$. By (3.6)

$$
|F|^p \leq (1 + \epsilon) \frac{(\lambda - \lambda)^p}{2^p} |u_x|^p + N(\epsilon) |f|^p.
$$

Thus, for sufficiently small $\epsilon$, (since $N(p) \searrow 1$ as $p \searrow 2$)

$$
\frac{N(p)}{\kappa^p} \frac{(\lambda - \lambda)^p}{2^p} = N(p)(1 + \epsilon) \frac{(\lambda - \lambda)^p}{(\lambda + \lambda)^p} < 1. \quad (3.8)
$$
Obviously the claims of the lemma follow from this. □

**Lemma 3.3.** Assume that for any solution \( u \in \mathcal{S}_{p,\theta_0}^1(\tau) \) of (1.1), we have estimate (2.19) for \( \theta = \theta_0 \), then there exists \( \chi = \chi(d, p, \theta_0, \lambda, \Lambda) > 0 \) such that for any \( \theta \in (\theta_0 - \chi, \theta_0 + \chi) \), estimate (2.19) holds whenever \( u \in \mathcal{S}_{p,\theta}(\tau) \) is a solution of (1.1).

**Proof.** The lemma is essentially proved in [6] for SPDEs with constant coefficients. By Lemma 2.3, \( u \in \mathcal{S}_{p,\theta}(\tau) \) if and only if \( v := M^{(\theta-\theta_0)/p} u \in \mathcal{S}_{p,\theta_0}^1(\tau) \) and the norms \( \|u\|_{\mathcal{S}_{p,\theta}(\tau)} \) and \( \|v\|_{\mathcal{S}_{p,\theta_0}^1(\tau)} \) are equivalent. Denote \( \varepsilon = (\theta - \theta_0)/p \) and observe that \( v \) satisfies

\[
dv = (D_i(a^{ij}v_{xj} + \beta^i v + \tilde{f}^i) + \tilde{b}^i v_{xj} + c v + \tilde{\tilde{f}})dt + (\nu^k v + M^\varepsilon g^k)dw^k_t,
\]

where

\[
\tilde{f}^i = M^\varepsilon f^i - \varepsilon a^{ij}M^{-1}v,
\]

\[
\tilde{\tilde{f}} = M^\varepsilon \tilde{f} - M^{-1}\varepsilon (\tilde{\beta}^i v + a^{ij}v_{xj} - a^{ij}v_{xj} + b^i v + M^\varepsilon f^i).
\]

By assumption (remember that \( Mb^i \) and \( M\tilde{b} \) are bounded),

\[
\|v\|_{\mathcal{S}_{p,\theta_0}^1(\tau)} \leq N(\|\tilde{f}^i\|_{L_{p,\theta_0}(\tau)} + \|M^\varepsilon \tilde{f}\|_{H^{-1}_{p,\theta_0}(\tau)} + \|M^\varepsilon u_0\|_{U_{p,\theta_0}})
\]

\[
\leq N(\|f^i\|_{L_{p,\theta}(\tau)} + \|Mf\|_{H^{-1}_{p,\theta}(\tau)} + \|u_0\|_{U_{p,\theta}})
\]

\[
+ N\varepsilon(\|M^{-1}v\|_{L_{p,\theta_0}(\tau)} + \|v_x\|_{L_{p,\theta_0}(\tau)}).
\]

Thus it is enough to take \( \varepsilon \) sufficiently small (see (2.8)). The lemma is proved. □

Now we come back to our proof. As usual we may assume \( \tau \equiv T \) (see [7]), and due to Lemma 3.3, without loss of generality we assume that \( \theta = d \).

Take \( p_0 \) from Lemma 3.2. The method of continuity shows that to prove the theorem it suffices to prove that if \( p \leq p_0 \), then (2.19) holds true given that a solution \( u \in \mathcal{S}_{p,d}^1(T) \) already exists.

**Step 1.** We assume that \( b^i = \tilde{b}^i = c = \nu^k = 0 \). By (2.8) (or see Lemma 1.3 (i) in [11])

\[
\|u_x\|_{H^{\gamma}_{p,\theta}} \sim \|M^{-1}u\|_{H^{\gamma+1}_{p,\theta}}.
\]

Thus we estimate \( \|u_x\|_{L_{p,d}(T)} \) instead of \( \|M^{-1}u\|_{H^{1}_{p,d}(T)} \). By Theorem 3.3 in [11] there exists a solution \( v \in \mathcal{S}_{p,d}^1(T) \) of

\[
dv = (\Delta v + \tilde{f}) dt, \quad v(0, \cdot) = u_0,
\]

and furthermore

\[
\|v_x\|_{L_{p,d}(T)} \leq N\|M\tilde{f}\|_{H^{-1}_{p,d}(T)} + N\|u_0\|_{U^{1}_{p,d}}.
\]

(3.9)
Observe that \( u - v \) satisfies
\[
d(u - v) = D_i (a^{ij}(u - v)_{x^j} + \tilde{f}^i) \, dt + g^k \, dw^k_t, \quad (u - v)(0, \cdot) = 0,
\]
where \( \tilde{f}^i = f^i + (a^{ij} - \delta^{ij}) v_{x^j} \). Therefore (2.19) follows from Lemma 3.2 and (3.9).

**Step 2** (general case). By the result of step 1,
\[
\| M^{-1} u \|_{H^1_p}(T) \leq N \| M b^i M^{-1} u + f^i \|_{L^p_d(T)} + N \| u_0 \|_{L^1_p(T)}
\]
\[
+ N \| M \tilde{b}^i u_{x^i} + M^2 c M^{-1} u + M \tilde{f}^i \|_{L^1_p(T)} + N \| M v M^{-1} u + g \|_{L^p_d(T)}
\]
\[
\leq N \beta (\| M^{-1} u \|_{L^p_d(T)} + \| u_x \|_{L^p_d(T)})
\]
\[
+ N \| u_0 \|_{L^1_p(T)} + N \| f^i \|_{L^p_d(T)} + N \| M \tilde{f}^i \|_{L^1_p(T)} + N \| g \|_{L^p_d(T)}.
\]
Now it is enough to choose \( \beta_0 \) such that for any \( \beta \leq \beta_0 \),
\[
N \beta (\| M^{-1} u \|_{L^p_d(T)} + \| u_x \|_{L^p_d(T)}) \leq 1/2 \| M^{-1} u \|_{L^1_p(T)}.
\]
The theorem is proved.

4. **Proof of Theorem 2.8**

First we need the following result on \( \mathbb{R}^d \) proved in [15].

**Lemma 4.1.** There exists \( p_0 = p_0(\lambda, \Lambda, d) > 2 \) such that if \( p \in [2, p_0) \)
and \( u \in \mathcal{H}^1_{p,0}(T) \) is a solution of
\[
du = D_i (a^{ij} u_{x^j} + f^i) dt + g^k dw^k_t, \quad (4.1)
\]
then
\[
\| u_x \|_{L^p(T)} \leq N (\| f \|_{L^p(T)} + \| g \|_{L^p(T)}).
\]

Again, to prove the theorem, we only show that the apriori estimate (2.20) holds for \( p < p_0 \) (also see step 1 below).

As in theorem 5.1 in [7], considering \( u - v \), where \( v \in \mathcal{H}^1_p(T) \) is the solution of
\[
dv = \Delta vd_t, \quad v(0, \cdot) = u_0,
\]
without loss of generality we assume that \( u(0, \cdot) = 0 \).

**Step 1.** Assume that \( b^i = \tilde{b}^i = c = \nu^k = 0 \). By Theorem 5.1 in [7],
there exists a solution \( v \in \mathcal{H}^1_{p,0}(T) \) of
\[
dv = (\Delta v + \tilde{f}) dt,
\]
and it satisfies
\[
\| v_x \|_{L^p(T)} \leq N \| \tilde{f} \|_{L^1_p(T)}.
\]
(4.2)

Observe that \( \tilde{u} := u - v \) satisfies
\[
d\tilde{u} = D_i (a^{ij} \tilde{u}_{x^j} + \tilde{f}^i) dt + g^k dw^k_t,
\]
where $\tilde{f}^i = f^i + (A - I)v_x$. Thus the estimate (2.20) follows from Lemma 4.1 and (4.2).

**Step 2.** We show that there exists $\epsilon_1 > 0$ such that if $T \leq \epsilon_1$, then all the assertions of the theorem hold true. Thus without loss of generality we assume that $T \leq 1$.

Note that $\bar{b}^i u_{x^i} \in L_p(T)$ since $u \in H^{1/2}(T)$, so by Theorem 5.1 in [7], there exists a unique solution $v \in H_{p,0}^2(T)$ of

$$dv = (\Delta v + \bar{b}^i u_{x^i})dt,$$

and $v$ satisfies

$$\|v\|_{H^2_p(T)} \leq N\|u_x\|_{L_p(T)}^p.$$

By (2.10),

$$\|v_x\|_{L_p(T)}^p \leq N\|v\|_{H^{1/2}_p(T)}^p \leq N(T)\|u_x\|_{L_p(T)}^p, \quad (4.3)$$

where $N(T) \to 0$ as $T \to 0$. Observe that $u - v$ satisfies

$$d(u - v) = (D_i(a^{ij}(u - v)_{x^i}) + (a^{ij} - \delta^{ij})v_x + b^i u + f^i) + cu + \tilde{f}) dt + (\nu^k u + g^k) dw^k.$$

By the result of step 1,

$$\|(u - v)_x\|_{L_p(T)} \leq N\|(a^{ij} - \delta^{ij})v_x + b^i u + f^i\|_{L_p(T)} + \|cu + \tilde{f}\|_{H^{-1}_p(T)} + \|\nu^k u + g\|_{L_p(T)}$$

$$\leq N(\|v_x\|_{L_p(T)} + \|f^i\|_{L_p(T)} + \|\tilde{f}\|_{H^{-1}_p(T)} + \|g\|_{L_p(T)} + \|u\|_{L_p(T)}),$$

where constants $N$ are independent of $T$ ($T \leq 1$). This and (4.3) yield

$$\|u_x\|_{L_p(T)} \leq NN(T)\|u_x\|_{L_p(T)} + N\|f^i\|_{L_p(T)} + N\|\tilde{f}\|_{H^{-1}_p(T)} + N\|g\|_{L_p(T)}$$

$$+ N\|\nu^k u + g\|_{L_p(T)}.$$

Note that the above inequality holds for all $t \leq T$. Choose $\epsilon_1$ so that $NN(T) \leq 1/2$ for all $T \leq \epsilon_1$, then for any $t \leq T \leq \epsilon_1$ (see Lemma 2.3),

$$\|u\|^p_{H^1_p(t)} \leq N\|u\|_{L_p(t)}^p + N(\|f^i\|_{H^{-1}_p(t)}^p + \|\tilde{f}\|_{L_p(t)}^p + \|g\|_{L_p(t)}^p)$$

$$\leq N\int_0^t \|u\|^p_{H^1_p(t)} dt + N(\|f^i\|_{H^{-1}_p(T)}^p + \|\tilde{f}\|_{L_p(T)}^p + \|g\|_{L_p(T)}^p).$$

Gronwall’s inequality leads to (2.20).

**Step 3.** Consider the case $T > \epsilon_1$. To proceed further, we need the following lemma.
Lemma 4.2. Let $\tau \leq T$ be a stopping and $du(t) = f(t)dt + g^k(t)dw^k_t$.

(i) Let $u \in H^\gamma_{p,0}(\tau)$. Then there exists a unique $\tilde{u} \in H^\gamma_{p,0}(T)$ such that $\tilde{u}(t) = u(t)$ for $t \leq \tau$ (a.s) and, on $(0, T)$,

$$d\tilde{u} = (\Delta \tilde{u}(t) + \tilde{f}(t))dt + g^kI_{t\leq\tau}dw^k_t, \tag{4.4}$$

where $\tilde{f} = (f(t) - \Delta u(t))I_{t\leq\tau}$. Furthermore,

$$\|\tilde{u}\|_{H^\gamma_{p,0}(T)} \leq N\|u\|_{H^\gamma_{p,0}(\tau)}, \tag{4.5}$$

where $N$ is independent of $u$ and $\tau$.

(ii) all the claims in (i) hold true if $u \in H^\gamma_{p,0}(G, \tau)$ and if one replace the space $H^\gamma_{p,0}(\tau)$ and $H^\gamma_{p,0}(T)$ with $H^\gamma_{p,0}(G, \tau)$ and $H^\gamma_{p,0}(G, T)$, respectively.

Proof. (i) Note $\tilde{f} \in H^\gamma_{p,0}(T), gI_{t\leq\tau} \in H^\gamma_{p,0}(T)$, so that, by Theorem 5.1 in [7], equation (4.4) has a unique solution $\tilde{u} \in H^\gamma_{p,0}(T)$ and (4.5) holds. To show that $\tilde{u}(t) = u(t)$ for $t \leq \tau$, notice that, for $t \leq \tau$, the function $v(t) = \tilde{u}(t) - u(t)$ satisfies the equation

$$v(t) = \int_0^t \Delta v(s)ds, \quad v(0, \cdot) = 0.$$

Theorem 5.1 in [7] shows that $v(t) = 0$ for $t \leq \tau$ (a.e).

(ii) It is enough to repeat the arguments in (i) using Theorem 2.9 in [1] (instead of Theorem 5.1 in [7]).

Now, to complete the proof, we repeat the arguments in [4]. Take an integer $M \geq 2$ such that $T/M \leq \varepsilon_1$, and denote $t_m = Tm/M$. Assume that, for $m = 1, 2, ..., M - 1$, we have the estimate (2.20) with $t_m$ in place of $\tau$ (and $N$ depending only on $d, p, \lambda, \Lambda, K$ and $T$). We are going to use the induction on $m$. Let $u_m \in H^\gamma_{p,0}$ be the continuation of $u$ on $[t_m, T]$, which exists by Lemma 4.2(i) with $\gamma = -1$ and $\tau = t_m$. Denote $v_m := u - u_m$, then (a.s) for any $t \in [t_m, T], \phi \in C^\infty(G)$ (since $du_m = \Delta u_m dt$ on $[t_m, T]$)

$$v_m(t, \phi) = -\int_{t_m}^t (a^{ij}v_{mx^i} + b^iv_m + f^i_{m, \phi_{x^i}})(s)ds$$

$$+ \int_{t_m}^t (\bar{b}^jv_{mx^i} + cu_m + f_m, \phi)(s)ds + \int_{t_m}^t (\nu^k v_m + g^k_m, \phi)(s)dw^k_s,$$

where

$$f^i_m = (a^{ij} - \delta^{ij})u_{mx^i} + b^i u_m + f^i, \quad f_m = \bar{b}^j u_{mx^i} + cu_m + \bar{f}, \quad g^k_m = \nu^k u_m + \bar{g}^k.$$
Next instead of random processes on $[0, T]$ one considers processes given on $[t_m, T]$ and, in a natural way, introduce spaces $\mathcal{H}_p^1([t_m, T])$, $L_p([t_m, T])$, $H_p^0([t_m, T])$. Then one gets a counterpart of the result of step 2 and concludes that

$$E\int_{t_m}^{t_{m+1}} \|(u - u_m)(s)\|_{H_p^1} ds \leq NE\int_{t_m}^{t_{m+1}} (\|f_m(s)\|_{L_p}^p + \|f_m(s)\|_{H_p^1}^p + \|g_m(s)\|_{L_p}^p) ds.$$

Thus by the induction hypothesis we conclude

$$E\int_0^{t_{m+1}} \|u(s)\|_{H_p^1}^p ds \leq NE\int_0^T \|u_m(s)\|_{H_p^1}^p ds$$

$$+ NE\int_{t_m}^{t_{m+1}} \|(u - u_m)(s)\|_{H_p^1}^p ds$$

$$\leq N(\|f^i\|_{L_p}^p(t_{m+1}) + \|\tilde{f}^i\|_{H_p^1(t_{m+1})}^p + \|g\|_{L_p}^p(t_{m+1})).$$

We see that the induction goes through and thus the theorem is proved.

### 5. Proof of Theorem 2.8

As usual we may assume $\tau \equiv T$. It is known (see [1]) that for any $u_0 \in U_{p, \theta}^1(G)$ and $(f, g) \in \psi^{-1}H_{p, \theta}^{-1}(G, T) \times L_{p, \theta}(G, T)$, there exists $u \in \mathcal{S}_{p, \theta}^1(G, T)$ such that $u(0, \cdot) = u_0$ and

$$du = (\Delta u + f) dt + g^k dw^k_t. \quad (5.1)$$

Thus as before, to finish the proof of the theorem, we only need to establish the apriori estimate (2.13) assuming that $u \in \mathcal{S}_{p, \theta}^1(G, T)$ satisfies (1.1) with initial data $u_0 = 0$, where $p \in [2, p_0)$ and $\theta \in (d - \chi, d + \chi)$.

To proceed we need the following results.

**Lemma 5.1.** Let $u \in \mathcal{S}_{p, \theta}^1(G, T)$ be a solution of (1.1). Then

(i) there exists $\varepsilon_0 \in (0, 1)$ (independent of $u$) such that if $u$ has support in $B_{\varepsilon_0}(x_0), x_0 \in \partial G$ then (2.13) holds.

(ii) if $u$ has support on $G_\varepsilon$ for some $\varepsilon > 0$, where $G_\varepsilon := \{x \in G : \text{dist}(x, \partial G) > \varepsilon\}$, then then (2.13) holds.

**Proof.** The second assertion of the lemma follows from Theorem 2.8 since in this case (see [12]) $u \in \mathcal{H}_p^1(T)$ and

$$\|u\|_{\mathcal{S}_{p, \theta}^1(G, T)} \sim \|u\|_{\mathcal{H}_p^1(T)}.$$

To prove the first assertion, we use Theorem 2.7. Let $x_0 \in \partial G$ and $\Psi$ be a function from Assumption 2.1. It is shown in [5] (or see [1]) that
\( \Psi \) can be chosen such that \( \Psi \) is infinitely differentiable in \( G \cap B_{m}(x_0) \) and satisfies
\[
[\Psi]_{n,B_{r_0}(x_0) \cap G}^{(0)} + [\Psi^{-1}]_{n,J_+}^{(0)} < N(n) < \infty
\] (5.2)
and
\[
\rho(x) \Psi_{xx}(x) \to 0 \quad \text{as} \quad x \in B_{r_0}(x_0) \cap G, \quad \text{and} \quad \rho(x) \to 0, \quad (5.3)
\]
where the constants \( N(n) \) and the convergence in (5.3) are independent of \( x_0 \).

Define \( r = r_0/K_0 \) and fix smooth functions \( \eta \in C^\infty_0(B_r), \varphi \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \eta, \varphi \leq 1 \), and \( \eta = 1 \) in \( B_{r/2}, \varphi(t) = 1 \) for \( t \leq -3, \) and \( \varphi(t) = 0 \) for \( t \geq -1 \) and \( 0 \geq \varphi' \geq -1 \). Observe that \( \Psi(B_{r_0}(x_0)) \) contains \( B_r \). For \( m = 1, 2, \ldots, t > 0, x \in \mathbb{R}_+^d \) define \( \varphi_m(x) = \varphi(m^{-1} \ln x^1) \).

Also we denote \( \Psi_i^j := D_i\Psi^j, \Psi_{rs}^j := D_rD_s\Psi^j, \Phi_r^j := D_r(\Psi_{rs}^j(\Psi^{-1}))(\Psi), \)
\[
\hat{a}_m := \hat{a}_m \eta(x) \varphi_m + (1 - \eta \varphi_m)I, \quad \hat{b}_m := \hat{b}_m \eta \varphi_m, \quad \hat{\nu}_m := \hat{\nu}_m \eta \varphi_m,
\]
where
\[
\hat{a}^ij(t, x) = \hat{a}^ij(t, \Psi^{-1}(x)), \quad \hat{b}^i(t, x) = \hat{b}^i(t, \Psi^{-1}(x)),
\]
\[
\hat{\nu}(t, x) = \nu(t, \Psi^{-1}(x)), \quad \hat{a}^ij = a^r\Psi^j_{xx} \Psi_{rs}^j, \quad \hat{b}^i = b^i \Psi_r^j,
\]
\[
\hat{\nu} = \hat{\nu} \Psi_r^j + a^r \Psi^j_{as} \Phi_r^j, \quad \hat{c} = c + b^r \Phi_r^j.
\]

Take \( \beta_0 \) from Theorem 2.7. Observe that \( \varphi(m^{-1} \ln x^1) = 0 \) for \( x^1 \geq e^{-m} \). Also we easily see that (5.3) implies \( x^1 \Psi_{xx}(\Psi^{-1}(x)) \to 0 \) as \( x^1 \to 0 \). Using these facts and Assumption 2.2(iv), one can find \( m > 0 \) independent of \( x_0 \) such that
\[
x^1|\hat{b}_m(t, x)| + x^1|\hat{b}_m(t, x)| + (x^1)^2|\hat{c}_m(t, x)| + x^1|\hat{\nu}_m(t, x)| \leq \beta_0,
\]
whenever \( t > 0, x \in \mathbb{R}_+^d \).

Now we fix a \( \varepsilon_0 < r_0 \) such that
\[
\Psi(B_{\varepsilon_0}(x_0)) \subset B_{r/2} \cap \{ x : x^1 \leq e^{-3m} \}.
\]

Let’s denote \( v := u(\Psi^{-1}) \) and continue \( v \) as zero in \( \mathbb{R}_+^d \setminus \Psi(B_{\varepsilon_0}(x_0)) \).

Since \( \eta \varphi_m = 1 \) on \( \Psi(B_{\varepsilon_0}(x_0)) \), the function \( v \) satisfies
\[
dv = \left( (\hat{a}^ij \varphi_{x^ix^j} + \hat{b}^i \varphi_{x^i} + \hat{f}^i) \varphi_{x^i} + \hat{b}_m \varphi_{x^i} + \hat{c}_m \varphi + \hat{\nu}\right) dt + (\hat{b}^k_m \varphi + \hat{g}^k) dw^k,
\]
where
\[
\hat{f}^i = f^i(\Psi^{-1}), \quad \hat{f} = f(\Psi^{-1}), \quad \hat{g}^k = g^k(\Psi^{-1}).
\]
Next we observe that by (5.2) and Theorem 3.2 in [12] (or see [5]) for any \( \nu, \alpha \in \mathbb{R} \) and \( h \in \psi^{-\alpha} H^\nu_{p, \theta}(G) \) with support in \( B_{\varepsilon_0}(x_0) \)

\[
\| \psi^\alpha h \|_{H^\nu_{p, \theta}(G)} \sim \| M^\alpha h(\Psi^{-1}) \|_{H^\nu_{p, \theta}}, \tag{5.4}
\]

Therefore we conclude that \( v \in \mathcal{F}^1_{p, \theta}(T) \). Also by Theorem 2.7 we have

\[
\| M^{-1} v \|_{\mathcal{F}^1_{p, \theta}(T)} \leq N \| \hat{f} \|_{L^p(T)} + N \| M \hat{f} \|_{L^1_{p, \theta}(T)} + N \| \hat{g} \|_{L^p(T)}.
\]

Finally (5.4) leads to (2.13). The lemma is proved.

Coming back to our proof, we choose a partition of unity \( \zeta_m, m = 0, 1, 2, \ldots, N_0 \) such that \( \zeta^0 \in C_0^\infty(G), \zeta^{(m)} = \zeta\left(\frac{2(x-x_m)}{\varepsilon_0}\right), \zeta \in C_0^\infty(B_1(0)) \), \( x_m \in \partial G, m \geq 1 \), and for any multi-indices \( \alpha \)

\[
\sup_x \sum \psi^{[\alpha]} |D^{\alpha} \zeta^{(m)}| < N(\alpha) < \infty, \tag{5.5}
\]

where the constant \( N(\alpha) \) is independent of \( \varepsilon_0 \) (see section 6.3 in [9]). Thus it follows (see [12]) that for any \( \nu \in \mathbb{R} \) and \( h \in H^\nu_{p, \theta}(G) \) there exist constants \( N \) depending only \( p, \theta, \nu \) and \( N(\alpha) \) (independent of \( \varepsilon_0 \)) such that

\[
\| h \|_{H^\nu_{p, \theta}(G)} \leq N \sum \| \zeta_m h \|_{H^\nu_{p, \theta}(G)} \leq N \| h \|_{H^\nu_{p, \theta}(G)}; \tag{5.6}
\]

\[
\sum \| \psi \zeta_m h \|_{H^\nu_{p, \theta}(G)} \leq N \| h \|_{H^\nu_{p, \theta}(G)}; \tag{5.7}
\]

Also,

\[
\sum \| \zeta^{(m)} h \|_{H^\nu_{p, \theta}(G)} \leq N(\varepsilon_0) \| h \|_{H^\nu_{p, \theta}(G)}; \tag{5.8}
\]

where the constant \( N(\varepsilon_0) \) depends also on \( \varepsilon_0 \).

Using the above inequalities and Lemma 5.1 we will show

\[
\| u_x \|_{L^p_{p, \theta}(G,t)} \leq N \| u \|_{L^p_{p, \theta}(G,t)} + \text{appropriate norms of } f^i, \tilde{f}, \tilde{g}, \tag{5.9}
\]

and we will drop the term \( \| u \|_{L^p_{p, \theta}(G,t)} \) using (2.9). But as one can see in (5.10) below, one has to handle the term \( \alpha^{ij} u_{x_j} \zeta^m \). Obviously if the right side of inequality (5.9) contains the norm \( \| u_x \|_{L^p_{p, \theta}(G,T)} \), then this is useless. The following arguments below are used just to avoid estimating \( \| \alpha^{ij} u_{x_j} \zeta^m \|_{L^p_{p, \theta}(G,T)} \).

Denote \( u^m = u^m \zeta^m, m = 0, 1, \ldots, N_0 \). Then \( u^m \) satisfies

\[
du^m = (D_i(a^{ij} u^m_{x_j} + b^i u^m + f^{m,i}) + \tilde{b}^i u^m + cu^m + \tilde{f}^m - a^{ij} u_{x_j} \zeta^m) \, dt + (\nu^k u^m + \zeta^m g^k) \, dw^k_t; \tag{5.10}
\]

where

\[
f^{m,i} = f^i \zeta - a^{ij} u_{x_j} \zeta^m, \]

\[
\tilde{f}^m = b^i u^m - f^i \zeta^m + \tilde{b}^i u_{x_j} \zeta^m + \tilde{f}^m \zeta^m.
\]
Since \( \psi^{-1}a^{ij}u_{x^j}\zeta^m \in \psi^{-1}\mathbb{L}_{p,\theta}(G, T) \), by Theorem 2.9 in [1] (or Theorem 2.10 in [5]), there exists unique solution \( v^m \in S^2_{p,\theta,0}(G, T) \) of
\[
dv = (\Delta v - \psi^{-1}a^{ij}u_{x^j}\zeta^m)dt,
\]
and furthermore
\[
\|v^m\|_{S^2_{p,\theta}(G, T)} \leq N\|a^{ij}u_{x^j}\zeta^m\|_{L_{p,\theta}(G, T)}. \tag{5.11}
\]
By (2.2) and Lemma 2.3,
\[
\|v^m\|_{L_{p,\theta}(G, T)} + \|\psi v^m\|_{L_{p,\theta}(G, T)} \leq N(T)\|a^{ij}u_{x^j}\zeta^m\|_{L_{p,\theta}(G, T)}, \tag{5.12}
\]
where \( N(T) \to 0 \) as \( T \to 0 \).

For \( m \geq 1 \), define \( \eta^m(x) = \zeta(\frac{x-x_0}{\epsilon_0}) \) and fix a smooth function \( \eta^0 \in C^\infty_0(G) \) such that \( \eta^0 = 1 \) on the support of \( \zeta^0 \). Now we denote \( \bar{u}^m := \psi v^m\eta^m \), then \( \bar{u}^m \in S^2_{p,\theta}(G, T) \) satisfies
\[
d\bar{u}^m = (\Delta \bar{u}^m + \tilde{f}^m - a^{ij}u_{x^j}\zeta^m)dt, \tag{5.13}
\]
where \( \tilde{f}^m = -2u^m(\eta^m\psi)_{x^1} - v^m\Delta(\eta^m\psi) \). Finally by considering \( \bar{u}^m := u^m - \bar{u}^m \) we can drop the term \( a^{ij}u_{x^j}\zeta^m \) in (5.10) because \( \bar{u}^m \) satisfies
\[
d\bar{u}^m = (D_i(a^{ij}\bar{u}_{x^j} + b^i\bar{u}^m + F^{m,i}) + \bar{f}^m)dt + (\nu^k\bar{u}^m + G^{m,k})du^k, \tag{5.14}
\]
where
\[
F^{m,i} = f^i\zeta^m - a^{ij}u_{x^j} + b^i\bar{u}^m + (a^{ij} - \delta^{ij})\bar{u}^m,
\]
\[
\tilde{F}^m = \bar{b}^i\bar{u}_{x^j} + cu^m - b^i\bar{u}\zeta^m + f^i\zeta^m - \bar{b}^i\bar{u}\zeta^m + \tilde{f}^m + 2v^m(\eta^m\psi)_{x^1} + v^m\Delta(\eta^m\psi),
\]
\[
G^{m,k} = \zeta^m g^k + \nu^k\bar{u}^m.
\]

By Lemma 5.1, for any \( t \leq T \),
\[
\|\psi^{-1}\bar{u}^m\|^p_{L^{p,\theta}(G, t)} \leq N\|F^{m,i}\|_{L_{p,\theta}(G, t)} + N\|\psi \tilde{F}^m\|_{L^{p,\theta-1}(G, t)} + N\|G^m\|_{L_{p,\theta}(G, t)}.
\]

Remember that \( \psi b^i, \psi \bar{b}^i, \psi^2c, \psi_x \) and \( \psi\psi_{xx} \) are bounded and \( \|\cdot\|_{L^{p,\theta}} \leq \|\cdot\|_{L_{p,\theta}} \). By (5.6),(5.7) and (5.8),
\[
\sum \|\psi \tilde{F}^m\|_{L^{p,\theta}(G, t)} \leq N\|\psi \tilde{F}\|_{L^{p,\theta-1}(G, t)} + \|f^i\|_{L_{p,\theta}(G, t)} + \|u\|_{L^{p,\theta}(G, t)}
\]
\[
+ N\sum (\|\bar{u}^m\|_{L_{p,\theta}(G, t)} + \|\psi^{-1}\bar{u}^m\|_{L^{p,\theta}(G, t)} + \|\psi v^m\|_{L_{p,\theta}(G, t)} + \|v^m\|_{L_{p,\theta}(G, t)})
\]
\[
\leq N\|\psi \tilde{F}\|_{L^{p,\theta-1}(G, t)} + \|f^i\|_{L_{p,\theta}(G, t)} + \|u\|_{L^{p,\theta}(G, t)} + \sum \|v^m\|_{L^{p,\theta}(G, t)}.
\]

Similarly (actually much easily) the sums
\[
\sum \|F^{m,i}\|_{L_{p,\theta}(G, t)} + \sum \|G^m\|_{L_{p,\theta}(G, t)}
\]
can be handled. Then one gets for each \( t \leq T \) (see (5.12) and note that \( \psi^{-1} \vec{u}^m = \nu^m \eta^m \)),

\[
\| \psi^{-1} u \|_{L^p_{\tilde{H}^1_p, \vartheta}(G, t)}^p \leq N \sum \| \psi^{-1} u_m \|_{L^p_{\tilde{H}^1_p, \vartheta}(G, t)}^p \\
\leq N \sum \| \psi^{-1} \vec{u}^m \|_{L^p_{\tilde{H}^1_p, \vartheta}(G, t)}^p + N \sum \| \nu^m \eta^m \|_{L^p_{\tilde{H}^1_p, \vartheta}(G, t)}^p \\
\leq N \| f^i \|_{L^p_{\vartheta}(G, T)} + N \| \psi \tilde{f} \|_{L^p_{\tilde{H}^1_p, \gamma}(G, T)} + N \| g \|_{L^p_{\vartheta}(G, t)} \\
+ N \| u \|_{L^p_{\vartheta}(G, t)} + NN(t) \| u_x \|_{L^p_{\vartheta}(G, t)}.
\]

Since \( \| u_x \|_{L^p_{\vartheta}} \leq N \| \psi^{-1} u \|_{L^p_{\tilde{H}^1_p, \vartheta}} \), we can choose \( \varepsilon_2 \in (0, 1] \) such that

\[
NN(t) \| u_x \|_{L^p_{\vartheta}(G, t)} \leq 1/2 \| \psi^{-1} u \|_{L^p_{\tilde{H}^1_p, \vartheta}(G, t)}, \text{ if } t \leq T \leq \varepsilon_2,
\]

and therefore

\[
\| u \|_{L^p_{\tilde{H}^1_p, \vartheta}(G, t)} \leq N \int_0^t \| u \|_{L^p_{\tilde{H}^1_p, \vartheta}(G, s)} ds + N \| f^i \|_{L^p_{\vartheta}(G, T)} \\
+ N \| \psi \tilde{f} \|_{L^p_{\tilde{H}^1_p, \gamma}(G, T)} + N \| g \|_{L^p_{\vartheta}(G, T)}.
\]

This and Gronwall’s inequality lead to (2.13) if \( T \leq \varepsilon_2 \). For the general case, one repeats step 3 in the proof of Theorem 2.8 using Lemma 4.2 (ii) instead of Lemma 4.2 (i). The theorem is proved.

References


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