Corrections to "On Lévy processes conditioned to stay positive"

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Abstract
We correct two errors of omission in our paper, [2].

Key words: Lévy process conditioned to stay positive, weak convergence, excursion measure.

AMS 2000 Subject Classification: Primary 60G51, 60G17.

Submitted to EJP on September 25, 2007, final version accepted December 18, 2007.
We would like to correct two errors of omission in our paper, [2]. The first occurs in equation (2.4), where we overlooked the possibility that the downgoing ladder time process has a positive drift. This happens if and only if 0 is not regular for \((0, \infty)\). If this drift is denoted by \(\eta\), the correct version of (2.4) is

\[
\mathbb{P}_x(\tau_{(-\infty,0)} > e/\varepsilon) = \mathbb{P}(X_{e/\varepsilon} \geq -x) = \mathbb{E} \left( \int_{[0,\infty)} e^{-\varepsilon t} 1_{(X_t \geq -x)} \, dL_t \right) [\eta\varepsilon + \eta(e/\varepsilon < \zeta)],
\]

and the correct version of (2.5) is:

\[
h(x) = \lim_{\varepsilon \to 0} \frac{\mathbb{P}_x(\tau_{(-\infty,0)} > e/\varepsilon)}{\eta\varepsilon + \eta(e/\varepsilon < \zeta)}.
\]

However this makes no essential difference to the proof of the following Lemma 1: we just need to replace \(\eta(e < \zeta)\) by \(\eta\varepsilon + \eta(e < \zeta)\) four times, and \(\eta(\zeta)\) by \(\eta + \eta(\zeta)\) in (2.6). The details can be seen in section 8.2 of [3]. We should also mention that (1) can be found in [1]: see equation (8), p 174.

The second omission is that we failed to give any proof of

Corollary 1. Assume that 0 is regular upwards. For any \(t > 0\) and for any \(\mathcal{F}_t\)-measurable, continuous and bounded functional \(F\),

\[
\eta(F, t < \zeta) = k \lim_{x \to 0} \mathbb{E}_x^1(h(X_t)^{-1} F).
\]

The clear implication from our paper is that this follows immediately from our main result, Theorem 2, but this overlooks the singularity at zero of the function \(1/h(x)\). Since this Corollary has been cited in a number of recent papers, we give here a full proof of it.

Proof. From (3.2) and Theorem 2 of [2] we see that, for any fixed \(\delta > 0, t > 0\),

\[
\eta(F, t < \zeta, X_t > \delta) = k \lim_{x \to 0} \mathbb{E}_x^1(h(X_t)^{-1} F, X_t > \delta),
\]

and in particular, taking \(F \equiv 1\),

\[
\eta(t < \zeta, X_t > \delta) = k \lim_{x \to 0} \mathbb{E}_x^1(h(X_t)^{-1}, X_t > \delta) = k \lim_{x \to 0} \mathbb{P}_x(X_t > \delta, \tau_{(-\infty,0)} > t)/h(x).
\]

Suppose we can show that

\[
\eta(t < \zeta) = k \lim_{x \to 0} \mathbb{P}_x(\tau_{(-\infty,0)} > t)/h(x).
\]

(3)

Then, by subtraction,

\[
\eta(t < \zeta, X_t \leq \delta) = k \lim_{x \to 0} \mathbb{P}_x(X_t \leq \delta, \tau_{(-\infty,0)} > t)/h(x) = k \lim_{x \to 0} \mathbb{E}_x^1(h(X_t)^{-1}, X_t \leq \delta).
\]
Since \( n(t < \zeta, X_t = 0) = 0 \), if \( K \) is an upper bound for \( F \), we also have
\[
\lim_{\delta \to 0} n(F, t < \zeta, X_t \leq \delta) \leq K \lim_{\delta \to 0} n(t < \zeta, X_t \leq \delta) = 0,
\]
and the required conclusion follows.

To prove (3) we start with (1), and, since we are assuming that 0 is regular upwards, the drift \( \eta \) in the downwards ladder time process is zero, so we can write it as
\[
\int_0^\infty e^{-\varepsilon t} P_x(\tau_{(-\infty, 0)} > t) dt = h^{(\varepsilon)}(x) \int_0^\infty e^{-\varepsilon t} n(\zeta > t) dt,
\]
where \( h^{(\varepsilon)}(x) = \mathbb{E}\left( \int_0^\infty e^{-\varepsilon t} 1_{X_t \geq -x} dL_t \right) \). We know \( 0 \leq h^{(\varepsilon)}(x) \leq h(x) \), so
\[
\int_0^\infty e^{-\varepsilon t} P_x(\tau_{(-\infty, 0)} > t) dt \leq h(x) \int_0^\infty e^{-\varepsilon t} n(\zeta > t) dt.
\]

But we also have
\[
\liminf_{x \to 0} \frac{P_x(\tau_{(-\infty, 0)} > t)}{h(x)} \geq \lim_{\delta \to 0} \liminf_{x \to 0} \frac{P_x(\tau_{(-\infty, 0)} > t, X_t > \delta)}{h(x)} = \lim_{\delta \to 0} n(\zeta > t, X_t > \delta) = \overline{n}(\zeta > t).
\]
Together, these prove that
\[
\lim_{x \to 0} \int_0^\infty e^{-\varepsilon t} \frac{P_x(\tau_{(-\infty, 0)} > t)}{h(x)} dt = \int_0^\infty e^{-\varepsilon t} \overline{n}(\zeta > t) dt.
\]
Thus the measure with density \( \frac{P_x(\tau_{(-\infty, 0)} > t)}{h(x)} \) converges weakly to the measure with the continuous density \( \overline{n}(\zeta > t) \). But if \( 0 < c < t \) are fixed we have
\[
\lim_{x \to 0} P_x(\tau_{(-\infty, 0)} > t)/h(x) \geq c^{-1} \lim_{x \to 0} \int_t^{t+c} P_x(\tau_{(-\infty, 0)} > s) ds/h(x)
= c^{-1} \int_t^{t+c} n(\zeta > s) ds \geq \overline{n}(\zeta > t + c),
\]
\[
\lim_{x \to 0} P_x(\tau_{(-\infty, 0)} > t)/h(x) \leq c^{-1} \lim_{x \to 0} \int_{t-c}^t P_x(\tau_{(-\infty, 0)} > s) ds/h(x)
= c^{-1} \int_{t-c}^t n(\zeta > s) ds \leq \overline{n}(\zeta > t - c),
\]
and letting \( c \downarrow 0 \) we conclude that (3) holds, and hence the Corollary.

\( \square \)

**Acknowledgement** We thank Erik Baurdoux and Mladen Savov for bringing these errors to our attention.
References

