DEVIATION INEQUALITIES AND MODERATE DEVIATIONS FOR
ESTIMATORS OF PARAMETERS IN AN ORNSTEIN-UHLENBECK
PROCESS WITH LINEAR DRIFT

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Abstract
Some deviation inequalities and moderate deviation principles for the maximum likelihood estimators of parameters in an Ornstein-Uhlenbeck process with linear drift are established by the logarithmic Sobolev inequality and the exponential martingale method.

1 Introduction and main results
1.1 Introduction
We consider the following Ornstein-Uhlenbeck process

\[ dX_t = (-\theta X_t + \gamma) dt + dW_t, \quad X_0 = x \] \hspace{1cm} (1.1)

where \( W \) is a standard Brownian motion and \( \theta, \gamma \) are unknown parameters with \( \theta \in (0, +\infty) \). We denote by \( P_{\theta,\gamma,x} \) the distribution of the solution of (1.1).

It is known that the maximum likelihood estimators (MLE) of the parameters \( \theta \) and \( \gamma \) are (cf.

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\(^1\)RESEARCH SUPPORTED BY THE NATIONAL NATURAL SCIENCE FOUNDATION OF CHINA (10871153)
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[15]

\[
\hat{\theta}_T = \frac{-T \int_0^T X_t dX_t + (X_T - x) \int_0^T X_t dt}{T \int_0^T X_t^2 dt - \left( \int_0^T X_t dt \right)^2}
\]  
(1.2)

\[
= \theta + \frac{W_T \hat{\mu}_T - \int_0^T X_t dW_t}{T \hat{\sigma}_T^2},
\]

\[
\hat{\gamma}_T = \frac{-\int_0^T X_t dX_t \int_0^T X_t dX_t + (X_T - x) \int_0^T X_t^2 dt}{T \int_0^T X_t^2 dt - \left( \int_0^T X_t dt \right)^2}
\]  
(1.3)

\[
= \gamma + \frac{W_T \hat{\mu}_T - \int_0^T X_t dW_t}{T \hat{\sigma}_T^2},
\]

where

\[
\hat{\mu}_T = \frac{1}{T} \int_0^T X_t dt, \quad \hat{\sigma}_T^2 = \frac{1}{T} \int_0^T X_t^2 dt - \hat{\mu}_T^2.
\]  
(1.4)

It is known that \( \hat{\theta}_T \) and \( \hat{\gamma}_T \) are consistent estimators of \( \theta \) and \( \gamma \) and have asymptotic normality (cf. [15]).

For \( \gamma = 0 \) case, Florens-Landais and Pham ([9]) calculated the Laplace functional of \( \int_0^T X_t dX_t, \int_0^T X_t^2 dt \) by Girsanov's formula and obtained large deviations for \( \hat{\theta}_T \) by Gärtner-Ellis theorem. Bercu and Rouault ([11]) presented a sharp large deviation for \( \hat{\theta}_T \). Lezaud ([14]) obtained the deviation inequality of quadratic functional of the classical OU processes. We refer to [8] and [11] for the moderate deviations of some non-linear functionals of moving average processes and diffusion processes. In this paper we use the logarithmic Sobolev inequality (LSI) to study the deviation inequalities and the moderate deviations of \( \hat{\theta}_T \) and \( \hat{\gamma}_T \) for \( \gamma \neq 0 \) case.

### 1.2 Main results

Throughout this paper, let \( \lambda_T, T \geq 1 \) be a positive sequence satisfying

\[
\lambda_T \to \infty, \quad \frac{\lambda_T}{\sqrt{T}} \to 0.
\]  
(1.5)

**Theorem 1.1.** There exist finite positive constants \( C_0, C_1, C_2 \) and \( C_3 \) such that for all \( r > 0 \) and all \( T \geq 1 \),

\[
P_{\theta, T, x} \left( |\hat{\theta}_T - \theta| \geq r \right) \leq C_0 \exp \left\{ -C_1 r T E_{\theta, T, x} \left( \hat{\sigma}_T^2 \right) \min \{ 1, C_2 r \} \right\}
\]

\[
+ C_0 \exp \left\{ -C_3 T E_{\theta, T, x} \left( \hat{\sigma}_T^2 \right) \right\}
\]

and

\[
P_{\theta, T, x} \left( |\hat{\gamma}_T - \gamma| \geq r \right) \leq C_0 \exp \left\{ -C_1 r T E_{\theta, T, x} \left( \hat{\sigma}_T^2 \right) \min \{ 1, C_2 r \} \right\}
\]

\[
+ C_0 \exp \left\{ -C_3 T E_{\theta, T, x} \left( \hat{\sigma}_T^2 \right) \right\}.
\]

**Remark 1.1.** In this theorem and the remainder of the paper, all the constants involved depend on \( \theta, \gamma \) and the initial point \( x \).
Theorem 1.2. (1) \( \left\{ P_{\theta, \gamma, x} \left( \sqrt{\frac{T}{\lambda_T}} (\hat{\theta}_T - \theta) \in \cdot \right), T \geq 1 \right\} \) satisfies the large deviation principle with speed \( \lambda_T \) and rate function \( I_1(u) = \frac{u^2}{4\theta} \), that is, for any closed set \( F \) in \( \mathbb{R} \),

\[
\limsup_{n \to \infty} \frac{1}{\lambda_T} \log P_{\theta, \gamma, x} \left( \sqrt{\frac{T}{\lambda_T}} (\hat{\theta}_T - \theta) \in F \right) \leq -\inf_{u \in F} \frac{u^2}{4\theta},
\]

and open set \( G \) in \( \mathbb{R} \),

\[
\liminf_{n \to \infty} \frac{1}{\lambda_T} \log P_{\theta, \gamma, x} \left( \sqrt{\frac{T}{\lambda_T}} (\hat{\theta}_T - \theta) \in G \right) \geq -\inf_{u \in G} \frac{u^2}{4\theta}.
\]

(2) \( \left\{ P_{\theta, \gamma, x} \left( \sqrt{\frac{T}{\lambda_T}} (\hat{\gamma}_T - \gamma) \in \cdot \right), T \geq 1 \right\} \) satisfies the large deviation principle with speed \( \lambda_T \) and rate function \( I_2(u) = \frac{\theta u^2}{2(\theta + 2\gamma^2)} \), that is, for any closed set \( F \) in \( \mathbb{R} \),

\[
\limsup_{n \to \infty} \frac{1}{\lambda_T} \log P_{\theta, \gamma, x} \left( \sqrt{\frac{T}{\lambda_T}} (\hat{\gamma}_T - \gamma) \in F \right) \leq -\inf_{u \in F} \frac{\theta u^2}{2(\theta + 2\gamma^2)},
\]

and open set \( G \) in \( \mathbb{R} \),

\[
\liminf_{n \to \infty} \frac{1}{\lambda_T} \log P_{\theta, \gamma, x} \left( \sqrt{\frac{T}{\lambda_T}} (\hat{\gamma}_T - \gamma) \in G \right) \geq -\inf_{u \in G} \frac{\theta u^2}{2(\theta + 2\gamma^2)}.
\]

In \( \gamma = 0 \) case, the deviation inequalities of quadratic functionals of the classical OU process are obtained in [14]. For the large deviations and the moderate deviations of \( \hat{\theta}_T \), we refer to [1], [9] and [11]. The proofs of Theorem 1.1 and Theorem 1.2 are based on the LSI with respect to \( L^2 \)-norm in the Wiener space and Herbst’s argument (cf. [10], [12]).

2 Deviation inequalities

In this section, we give some deviation inequalities for the estimators \( \hat{\theta}_T \) and \( \hat{\gamma}_T \) by the logarithmic Sobolev inequality and the exponential martingale method. For deviation bounds for additive functionals of Markov processes, we refer to [3] and [18].

2.1 Moments

It is known that the solution of equation (1.1) has the following expression:

\[
X_t = \left( x - \frac{\gamma}{\theta} \right) e^{-\theta t} + \frac{\gamma}{\theta} + e^{-\theta t} \int_0^t e^{\theta s} dW_s. \tag{2.1}
\]

From this expression, it is easily seen that for any \( t \geq 0 \),

\[
\mu_t := E_{\theta, \gamma, x}(X_t) = \left( x - \frac{\gamma}{\theta} \right) e^{-\theta t} + \frac{\gamma}{\theta}, \tag{2.2}
\]

\[
\sigma^2_t := \text{Var}_{\theta, \gamma, x}(X_t) = \frac{1}{2\theta} (1 - e^{-2\theta t}) \tag{2.3}
\]
and for any $0 \leq s \leq t$,
\[ \text{Cov}_{\theta,\gamma,x}(X_s, X_t) = \frac{1}{2\theta} (1 - e^{-2\theta s}) e^{-\theta (t-s)}. \] (2.4)

Therefore
\[ E_{\theta,\gamma,x}(\hat{\alpha}_T) = \frac{1}{T} E_{\theta,\gamma,x} \left( \int_0^T X_t \, dt \right) = \frac{1}{2\theta T} \left( x - \frac{\gamma}{\theta} \right) (1 - e^{-\theta T}) + \frac{\gamma}{\theta}, \] (2.5)

\[ \text{Var}_{\theta,\gamma,x}(\hat{\mu}_T) = \frac{1}{T^2} E_{\theta,\gamma,x} \left( \left( \int_0^T e^{-\theta t} \int_0^t e^{\theta s} dW_s \, dt \right)^2 \right) \] (2.6)

\[ = \frac{1}{\theta^2 T^2} \left( T - \frac{1}{2\theta} (e^{-\theta T} - 1) + \frac{2}{\theta} (e^{-\theta T} - 1) \right) \]

and so for all $T \geq 1$,
\[ \text{Var}_{\theta,\gamma,x}(\hat{\mu}_T) \leq \frac{1}{2\theta^3 T} (2\theta + 1) \] (2.7)

and
\[ E_{\theta,\gamma,x}(\hat{\sigma}_T^2) = \frac{1}{2\theta} + \frac{1}{4\theta^2 T} (1 - e^{-2\theta T}) \left( -1 + 2\theta \left( x - \frac{\gamma}{\theta} \right)^2 \right) \]

\[ - \frac{1}{\theta^2 T^2} (1 - e^{-\theta T})^2 \left( x - \frac{\gamma}{\theta} \right)^2 (1 - e^{-\theta T}) \]

\[ - \frac{1}{\theta^2 T^2} \left( T - \frac{1}{2\theta} (e^{-\theta T} - 1) + \frac{2}{\theta} (e^{-\theta T} - 1) \right) \]

which implies
\[ \left| E_{\theta,\gamma,x}(\hat{\sigma}_T^2) - \frac{1}{2\theta} \right| \leq \frac{1}{\theta^2 T} \left( \theta \left( x - \frac{\gamma}{\theta} \right)^2 + \frac{2}{\theta} \right). \] (2.8)

Lemma 2.1. For any $0 \leq \alpha \leq \theta^2/4$, for all $T \geq 1$,
\[ E_{\theta,\gamma,x} \left( \exp \left( \alpha \int_0^T X_t^2 \, dt \right) \right) < \infty, \]

and there exist finite positive constants $L_1$ and $L_2$ such that for all $0 \leq \alpha \leq \theta^2/4$ and $T \geq 1$,
\[ E_{\theta,\gamma,x} \left( \exp \left( \alpha \int_0^T X_t^2 \, dt \right) \right) \leq L_1 e^{L_2 \alpha T}. \]

Proof. For any $0 \leq \alpha \leq \theta^2/4$, set $\kappa = \sqrt{\theta^2 - 2\alpha}$. Then by Girsanov theorem, we have
\[ \frac{dP_{\theta,\gamma,x}}{dP_{\kappa,\gamma,x}} = \exp \left\{ - \int_0^T (\theta - \kappa) X_t dX_t - \int_0^T (\alpha X_t^2 - \gamma(\theta - \kappa) X_t) dt \right\} \]
and so
\[
E_{\theta,T,x}\left(\exp\left(\alpha \int_0^T X_t^2 dt\right)\right)
\]
\[= E_{\kappa,T,x}\left(\frac{dP_{\theta,T,x}}{dP_{\kappa,T,x}} \exp\left(\alpha \int_0^T X_t^2 dt\right)\right)
\]
\[= E_{\kappa,T,x}\left(\exp\left(-\theta + \kappa\right) \int_0^T X_t dX_t + \gamma \int_0^T (\theta - \kappa) X_t dt\right)
\]
\[= E_{\kappa,T,x}\left(\exp\left(-\frac{(\theta - \kappa)}{2}\left(X_T^2 - T\right) + \gamma \int_0^T (\theta - \kappa) X_t dt\right)\right)
\]
\[\leq \exp\left(\frac{(\theta - \kappa)T}{2}\right) E_{\kappa,T,x}\left(\exp\left(\gamma \int_0^T (\theta - \kappa) X_t dt\right)\right)
\]
where the last inequality is due to \(\theta \geq \kappa\). Now we have to estimate \(E_{\kappa,T,x}(\exp\{\gamma \int_0^T (\theta - \kappa) X_t dt\})\).
Since under \(P_{\kappa,T,x}\),
\[
\bar{\mu}_T \sim N\left(\frac{1}{\kappa T}(x - \frac{\gamma}{\kappa}(1 - e^{-\kappa T}) + \frac{\gamma}{\kappa}) T - \frac{1}{2\kappa}(e^{-2\kappa T} - 1) + \frac{\gamma}{\kappa} e^{-\kappa T} - 1\right),
\]
we have
\[
E_{\kappa,T,x}\left(\exp\left(\frac{\gamma (\theta - \kappa)}{\kappa} \left(\left(x - \frac{\gamma}{\kappa}\right) (1 - e^{-\kappa T}) + \gamma T\right)\right)\right)
\]
\[= \exp\left(\frac{\gamma (\theta - \kappa)}{\kappa} \left(\left(x - \frac{\gamma}{\kappa}\right) (1 - e^{-\kappa T}) + \gamma T\right)\right)
\]
\[\cdot \exp\left(-\frac{\gamma^2 (\theta - \kappa)^2}{2\kappa^2} \left(T - \frac{1}{2\kappa}(e^{-2\kappa T} - 1) + \frac{\gamma}{\kappa} e^{-\kappa T} - 1\right)\right).
\]
Noting \(\theta / \sqrt{2} \leq \kappa \leq \theta, 0 \leq \theta - \kappa = 2\alpha/(\theta + \kappa) \leq 2\alpha/\theta\) and \((\theta - \kappa)^2 \leq \alpha \theta\) for all \(0 \leq \alpha \leq \theta^2/4\),
we complete the proof of the lemma.

\[\square\]

### 2.2 Logarithmic Sobolev inequality

Since the LSI with respect to the Cameron-Martin metric does not produce the concentration inequality of correct order in large time \(T\) for the functionals
\[
F(X) := \frac{1}{\sqrt{T}} \left(\int_0^T g(X_s) ds - \mathbb{E}\left(\int_0^T g(X_s) ds\right)\right),
\]
in order to get the concentration inequality of correct order for the functionals \(F(X)_n\), as pointed out by Djellout, Guillin and Wu (\cite{71}) we should establish the LSI with respect to the \(L^2\)-metric.
Let us introduce the logarithmic Sobolev inequality on \(W\) with respect to the gradient in \(L^2([0,T],\mathbb{R})\) (\cite{10}). Let \(\mu\) be the Wiener measure on \(W = C([0,T],\mathbb{R})\). A function \(f : W \to \mathbb{R}\) is said to be
differentiable with respect to the $L^2$-norm, if it can be extend to $L^2([0, T], \mathbb{R})$ and for any $w \in W$, there exists a bounded linear operator $Df(w) : g \to D_g f(w)$ on $L^2([0, T], \mathbb{R})$ such that

$$\lim_{\|g\|_2 \to 0} \frac{|f(w + g) - f(w) - D_g f(w)|}{\|g\|_2} = 0.$$ 

If $f : W \to \mathbb{R}$ is differentiable with respect to the $L^2$-norm, then there exists a unique element $\nabla f(w) = (\nabla_i f(w), t \in [0, T])$ in $L^2([0, T], \mathbb{R})$ such that

$$D_g f(w) = (\nabla f(w), g)_{L^2}, \text{ for all } g \in L^2([0, T], \mathbb{R}).$$

Denote by $C_b^1(W/L^2)$ the space of all bounded function $f$ on $W$, differentiable with respect to the $L^2$-norm, such that $\nabla f$ is also continuous and bounded from $W$ equipped with $L^2$-norm to $L^2([0, T], \mathbb{R})$. Applying Theorem 2.3 in [10] to the Ornstein-Uhlenbeck process with linear drift, we have

$$\text{Ent}_{\theta, \gamma, x}(f^2) \leq \frac{2}{\theta^2} E_{\theta, \gamma, x} \left( \int_0^T |\nabla_i f|^2 dt \right), \quad f \in C_b^1(W/L^2) \quad (2.9)$$

where the entropy of $f^2$ is given by

$$\text{Ent}_{\theta, \gamma, x}(f^2) = E_{\theta, \gamma, x}(f^2 \log f^2) - E_{\theta, \gamma, x}(f^2) \log E_{\theta, \gamma, x}(f^2).$$

**Lemma 2.2.** For any $|a| \leq \theta^2/4$,

$$E_{\theta, \gamma, x} \left( \exp \left\{ \alpha \left( \int_0^T X_i^2 dt - E_{\theta, \gamma, x} \left( \int_0^T X_i^2 dt \right) \right) \right\} \right) \leq E_{\theta, \gamma, x} \left( \exp \left\{ \frac{4\alpha^2}{\theta^2} \int_0^T X_i^2 dt \right\} \right)$$

and

$$E_{\theta, \gamma, x} \left( \exp \left\{ \alpha T \left( \mu_i^2 - E_{\theta, \gamma, x}(\mu_i^2) \right) \right\} \right) \leq E_{\theta, \gamma, x} \left( \exp \left\{ \frac{4\alpha^2}{\theta^2} \int_0^T X_i^2 dt \right\} \right).$$

**Proof.** We apply Theorem 2.7 in [12] to prove the conclusions of the lemma. Take $\mathcal{A}_1 = \{ af ; |a| \leq \theta^2/4 \}$ and $\mathcal{A}_2 = \{ ah ; |a| \leq \theta^2/4 \}$, where

$$f(w) = \int_0^T w_i^2 dt, \quad h(w) = \frac{1}{T} \left( \int_0^T w_i dt \right)^2.$$ 

Define

$$\Gamma_1(g_1) = \frac{4}{\theta^2} g_1^2, \quad g_1 \in \mathcal{A}_1; \quad \Gamma_2(g_2) = \frac{4}{\theta^2} g_2^2, \quad g_2 \in \mathcal{A}_2.$$

Then for any $\lambda \in [-1, 1]$, $g_1 \in \mathcal{A}_1$ and $g_2 \in \mathcal{A}_2$, $\lambda g_1 \in \mathcal{A}_1$, $\lambda g_2 \in \mathcal{A}_2$, $\Gamma_1(\lambda g_1) = \lambda^2 \Gamma_1(g_1)$, $\Gamma_2(\lambda g_2) = \lambda^2 \Gamma_2(g_2)$ and by Lemma 2.1

$$E_{\theta, \gamma, x} \left( \exp \{ \lambda \Gamma_1(g_1) \} \right) < \infty, \quad E_{\theta, \gamma, x} \left( \exp \{ \lambda \Gamma_2(g_2) \} \right) < \infty.$$ 

Choose a sequence of real $C^\infty$-functions $\Phi_n, n \geq 1$ with compact support such that $\lim_{n \to \infty} \sup_{|x| \leq M} |\Phi_n(x) - e^x| = 0$ for all $M \in (0, \infty)$. For any $g_1 = af \in \mathcal{A}_1$ and $g_2 = ah \in \mathcal{A}_2$, set

$$F_n(w) = \Phi_n \left( g_1(w)/2 \right), \quad H_n(w) = \Phi_n \left( g_2(w)/2 \right).$$
Then for any $g \in L^2([0, T], \mathbb{R})$,
\[
\lim_{\|g\|_2 \to 0} \frac{|F_n(w + g) - F_n(w) - \alpha \Phi_n'(g_1(w)/2) \langle w, g \rangle_{L^2}|}{\|g\|_2} = 0
\]
and
\[
\lim_{\|g\|_2 \to 0} \frac{|H_n(w + g) - H_n(w) - \alpha \Phi_n'(g_2(w)/2) \frac{1}{T} \int_0^T w_t dt |}{\|g\|_2} = 0.
\]
Therefore, $F_n, H_n \in C_0^1(W/L^2)$, $\nabla F_n = \alpha \Phi_n'(g_1(w)/2) w$, and
\[
\nabla H_n = \frac{\alpha}{T} \int_0^T w_t dt \Phi_n'(g_2(w)/2)
\]
and so by (2.9), we have
\[
\text{Ent}_{P_{\theta, x}}\left(F_n^2\right) \leq \frac{2}{\theta^2} E_{\theta, T, x} \left(\int_0^T |aw_t|^2 dt \left(\Phi_n'(g_1(w)/2)\right)^2\right)
\]
and
\[
\text{Ent}_{P_{\theta, x}}\left(H_n^2\right) \leq \frac{2}{\theta^2} E_{\theta, T, x} \left(\frac{1}{T} \left(\alpha \int_0^T w_t dt\right)^2 \left(\Phi_n'(g_2(w)/2)\right)^2\right).
\]
Letting $n \to \infty$ and by Lemma 2.1, we get
\[
\text{Ent}_{P_{\theta, x}}\left(e^{\epsilon_1}\right) \leq \frac{1}{2} E_{\theta, T, x} \left(\Gamma_1(g_1)e^{\epsilon_1}\right), \quad \text{Ent}_{P_{\theta, x}}\left(e^{\epsilon_2}\right) \leq \frac{1}{2} E_{\theta, T, x} \left(\Gamma_2(g_2)e^{\epsilon_2}\right), \quad (2.10)
\]
and so the conclusions of the lemma hold by Theorem 2.7 in [12] and $T \bar{\mu}_T^2 \leq \int_0^T X_t^2 dt$. \hfill \Box

### 2.3 Deviation inequalities

Since $X_T \sim N\left(\mu_T, \sigma_T^2\right)$, and under $P_{\theta, x}$
\[
\bar{\mu}_T \sim N \left(\frac{1}{\theta T} (x - \frac{\gamma}{\theta})(1 - e^{-\theta T}) + \frac{\gamma}{\theta^2 T^2} \left(T - \frac{1}{2\theta}(e^{-2\theta T} - 1) + \frac{2}{\theta^2 T}(e^{-\theta T} - 1)\right)\right),
\]
it is easily to get from Chebyshev inequality, for any $r > 0$,
\[
P_{\theta, x} \left(|X_T - E_{\theta, x}(X_T)| \geq r\right) \leq 2 \exp\left\{-\theta r^2\right\}, \quad (2.11)
\]
\[
P_{\theta, x} \left(|\bar{\mu}_T - E_{\theta, x}(\bar{\mu}_T)| \geq r\right) \leq 2 \exp\left\{-\frac{\theta^3 T^2 r^2}{2\theta + 1}\right\} \quad (2.12)
\]
where we used (2.7).
Lemma 2.3. There exist finite positive constants $C_0, C_1, C_2$ such that for all $r > 0$ and all $T \geq 1$,
\[
P_{\theta, y, x} \left( \left| \int_0^T X_t^2 dt - E_{\theta, y, x} \left( \int_0^T X_t^2 dt \right) \right| \geq rT \right) \leq C_0 \exp \left\{ -C_1 rT \min \{1, C_2 r\} \right\}
\]
and
\[
P_{\theta, y, x} \left( \left| \hat{\mu}_T^2 - E_{\theta, y, x}(\hat{\mu}_T^2) \right| \geq r \right) \leq C_0 \exp \left\{ -C_1 rT \min \{1, C_2 r\} \right\}.
\]
In particular, there exist finite positive constants $C_0, C_1, C_2$ such that for all $r > 0$ and all $T \geq 1$,
\[
P_{\theta, y, x} \left( \left| \hat{\sigma}_T^2 - E_{\theta, y, x}(\hat{\sigma}_T^2) \right| \geq r \right) \leq C_0 \exp \left\{ -C_1 rT \min \{1, C_2 r\} \right\}.
\]

Proof. We only prove the first inequality. By Lemma 2.2 and Lemma 2.1 there exist finite positive constants $L_1$ and $L_2$ such that for all $T \geq 1$, for any $|a| \leq \theta^2/4$,
\[
E_{\theta, y, x} \left( \exp \left\{ a \left( \int_0^T X_t^2 dt - E_{\theta, y, x} \left( \int_0^T X_t^2 dt \right) \right) \right\} \right) \leq L_1 e^{L_2 a^2 T}.
\]
Therefore, by Chebyshev inequality, for any $r > 0$, $T \geq 1$ and $|a| \leq \theta^2/4$,
\[
P_{\theta, y, x} \left( \int_0^T X_t^2 dt - E_{\theta, y, x} \left( \int_0^T X_t^2 dt \right) \geq rT \right) \leq L_1 e^{- \left( ar - L_2 a^2 \right) T}
\]
and
\[
P_{\theta, y, x} \left( \int_0^T X_t^2 dt - E_{\theta, y, x} \left( \int_0^T X_t^2 dt \right) \leq -rT \right) \leq L_1 e^{- \left( ar - L_2 a^2 \right) T}.
\]
Now, by
\[
\sup_{|a| \leq \theta^2/4} \{ ar - L_2 a^2 \} \geq \frac{\theta^2 r}{8} \min \left\{ 1, \frac{2r}{L_2 \theta^2} \right\},
\]
we obtain the first inequality of the lemma from the above estimates.

Lemma 2.4. There exist finite positive constants $C_0, C_1$ and $C_2$ such that for all $r > 0$ and all $T \geq 1$,
\[
P_{\theta, y, x} \left( \left| W_T \left( \hat{\mu}_T - \frac{y}{\theta} \right) \right| \geq rT \right) \leq C_0 \exp \left\{ -C_1 rT \min \{1, C_2 r\} \right\}.
\]

Proof. Since for any $r > 0$ and $T \geq 1$,
\[
\left\{ \left| W_T \left( \hat{\mu}_T - \frac{y}{\theta} \right) \right| \geq rT \right\} \subset \left\{ \left| W_T(\hat{\mu}_T - E_{\theta, y, x}(\hat{\mu}_T)) \right| \geq rT/2 \right\} \cup \left\{ \left| W_T \left( E_{\theta, y, x}(\hat{\mu}_T) - \frac{y}{\theta} \right) \right| \geq rT/2 \right\}
\]
\[
\subset \left\{ \left| W_T \right| \geq \sqrt{rT/2} \right\} \cup \left\{ \left| \hat{\mu}_T - E_{\theta, y, x}(\hat{\mu}_T) \right| \geq \sqrt{r} \right\} \cup \left\{ \left| W_T \right| \geq \frac{\theta rT}{2|\hat{\theta} - x|} \right\}
\]
by \((2.12)\) and \(W_t \sim N(0, T)\), we get
\[
P_{\theta,r,x} \left( \left| W_T (\hat{\mu}_r - \frac{Y}{\theta}) \right| \geq rT \right)
\leq 2 \exp \left( -\frac{Tr}{8} \right) + 2 \exp \left( -\frac{\theta^3 Tr}{2\theta + 1} \right) + 2 \exp \left( -\frac{\theta^2 r^2 T}{8(x - \frac{y}{\theta})^2} \right).
\]

**Lemma 2.5.** For each \(\beta \in \mathbb{R}\) fixed, there exist finite positive constants \(C_0, C_1, C_2\) such that for all \(r > 0\) and all \(T \geq 1\),
\[
P_{\theta,r,x} \left( \left| \int_0^T (X_t - \beta) \, dW_t \right| \geq rT \right) \leq C_0 \exp \{ -C_1 rT \min \{1, C_2 r\} \}.
\]

**Proof.** It is known that for \(\alpha \in \mathbb{R}\),
\[
M^{(\alpha)}_T = \exp \left\{ \alpha \int_0^T (X_t - \beta) \, dW_t - \frac{\alpha^2}{2} \int_0^T (X_t - \beta)^2 \, dt \right\}, \quad T \geq 0
\]
is \(\mathcal{F}_T\)-martingale, where \(\mathcal{F}_T := \sigma(W_t, t \leq T)\). Therefore, by Hölder inequality, we can get that for any \(\epsilon \in (0, 1]\),
\[
E_{\theta,\gamma,x} \left( \exp \left\{ \alpha \int_0^T (X_t - \beta) \, dW_t \right\} \right) \\
\leq \left( E_{\theta,\gamma,x} \left( \exp \left\{ \frac{(1 + \epsilon)^2 \alpha^2}{2 \epsilon} \int_0^T (X_t - \beta)^2 \, dt \right\} \right) \right)^{\frac{1}{1+\epsilon}} \left( E_{\theta,\gamma,x} \left( M_T^{(1+\epsilon)\alpha} \right) \right)^{\frac{1}{1+\epsilon}}.
\]

In particular, take \(\epsilon = 1\), then by Lemma \(2.1\) there exists finite positive constants \(L_1 = L_1(\theta, \beta, \gamma, x)\) and \(L_2 = L_2(\theta, \beta, \gamma, x)\) such that for all \(T \geq 1\), for any \(\alpha^2 \leq \theta^2 / 16\), by Cauchy-Schwartz inequality,
\[
E_{\theta,\gamma,x} \left( \exp \left\{ \alpha \int_0^T (X_t - \beta) \, dW_t \right\} \right) \\
\leq \left( E_{\theta,\gamma,x} \left( \exp \left\{ 2\alpha^2 \int_0^T (X_t - \beta)^2 \, dt \right\} \right) \right)^{\frac{1}{2}} \\
\leq \left( E_{\theta,\gamma,x} \left( \exp \left\{ 4\alpha^2 \int_0^T X_t^2 \, dt \right\} \right) \right)^{\frac{1}{2}} \left( E_{\theta,\gamma,x} \left( \exp \left\{ 4\alpha^2 \int_0^T (-2\beta X_t + \beta^2) \, dt \right\} \right) \right)^{\frac{1}{2}} \\
\leq L_1 e^{2\alpha^2 T}.
\]

Therefore, by Chebyshev inequality, the conclusion of the lemma holds. \(\square\)
Proof of Theorem 1.1

We only show the first inequality. The second one is similar. By

\[ \hat{\theta} - \theta = \frac{W_T \left( \hat{\mu} - \frac{\gamma}{\theta} \right) - \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t}{T \hat{\sigma}^2_T} \]

for any \( r > 0 \) and \( T \geq 1 \),

\[
p_{\theta, y, x} \left( |\hat{\theta} - \theta| \geq r \right) \\
\leq p_{\theta, y, x} \left( |\hat{\sigma}^2_T - E_{\theta, y, x}(\hat{\sigma}^2_T)| \geq E_{\theta, y, x}(\hat{\sigma}^2_T)/2 \right) \\
+ p_{\theta, y, x} \left( |W_T \left( \hat{\mu} - \frac{\gamma}{\theta} \right) - \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t| \geq E_{\theta, y, x}(\hat{\sigma}^2_T)rT/2 \right)
\]

Therefore, by Lemmas 2.3, 2.4 and 2.5 we obtain the first inequality of the theorem. \( \square \)

3 Moderate deviations

In this section, we show Theorem 1.1. By (1.2) and (1.3), we have the following estimates

\[ \left| (\hat{\theta} - \theta) + \frac{2\theta}{T} \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right| \]

\[
\leq \frac{|W_T \left( \hat{\mu} - \frac{\gamma}{\theta} \right)|}{T \hat{\sigma}^2_T} + \frac{|2\theta \hat{\sigma}^2_T - 1|}{T \hat{\sigma}^2_T} \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t
\]

and for

\[ \left| (\gamma - \gamma) - \frac{W_T}{T} + \frac{2\gamma}{T} \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right| \]

\[
\leq \frac{|\hat{\mu}||W_T \left( \hat{\mu} - \frac{\gamma}{\theta} \right)|}{T \hat{\sigma}^2_T} + \frac{|2\gamma \hat{\sigma}^2_T - \hat{\mu}|}{T \hat{\sigma}^2_T} \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t.
\]

Lemma 3.1. (1) For any \( r > 0 \),

\[
\lim_{T \to \infty} \log P_{\theta, y, x} \left( \left| X_t - \frac{\gamma}{\theta} \right| dW_t \geq \sqrt{T\lambda_T r} \right) = -\infty,
\]

\[
\lim_{T \to \infty} \frac{1}{\lambda_T} \log P_{\theta, y, x} \left( \left| X_t - \frac{\gamma}{\theta} \right| \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \geq \sqrt{T\lambda_T r} \right) = -\infty
\]

and

\[
\lim_{T \to \infty} \frac{1}{\lambda_T} \log P_{\theta, y, x} \left( \left| \frac{\sigma^2_T}{\lambda_T} - \frac{1}{2\theta} \right| \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \geq \sqrt{T\lambda_T r} \right) = -\infty.
\]
(2). For any $\delta > 0$,
\[
\limsup_{T \to \infty} \frac{1}{\lambda_T} \log P_{\theta, \gamma, x} \left( \left| (\hat{\theta}_T - \theta) - \frac{2\theta}{T} \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \delta \sqrt{\frac{\lambda_T}{T}} \right) = -\infty
\]
and
\[
\limsup_{T \to \infty} \frac{1}{\lambda_T} \log P_{\hat{\theta}, \gamma, x} \left( \left| (\hat{\gamma}_T - \gamma) - \frac{W_T}{T} - \frac{2\gamma}{T} \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \delta \sqrt{\frac{\lambda_T}{T}} \right) = -\infty.
\]

Proof. (1). We only give the proof of the third assertion in (1). The rest is similar. For any $L > 0$,
\[
\left\{ \left| \hat{\sigma}_T^2 - \frac{1}{2\theta} \right| \left| \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \sqrt{T \lambda_T} r \right\}
\]
\[
\subset \left\{ \left| \hat{\sigma}_T^2 - \frac{1}{2\theta} \right| \geq \frac{r}{L} \right\} \cup \left\{ \frac{1}{\sqrt{T \lambda_T}} \left| \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq L \right\}.
\]

By Lemma 2.3 and Lemma 2.5 we have
\[
\limsup_{T \to \infty} \frac{1}{\lambda_T} \log P_{\theta, \gamma, x} \left( \left| \hat{\sigma}_T^2 - \frac{1}{2\theta} \right| \geq \frac{r}{L} \right) = -\infty
\]
and
\[
\limsup_{T \to \infty} \frac{1}{\lambda_T} \log P_{\theta, \gamma, x} \left( \frac{1}{\sqrt{T \lambda_T}} \left| \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq L \right) \leq -L^2 C_1 C_2.
\]

Hence,
\[
\limsup_{T \to \infty} \frac{1}{\lambda_T} \log P_{\theta, \gamma, x} \left( \left| \hat{\sigma}_T^2 - \frac{1}{2\theta} \right| \left| \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \sqrt{T \lambda_T} r \right) \leq -L^2 C_1 C_2.
\]

Letting $L \to \infty$, we obtain the third conclusion.

(2). It follows from (3.1) and (3.2) that
\[
\left( \left| (\hat{\theta}_T - \theta) - \frac{2\theta}{T} \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \delta \sqrt{\frac{\lambda_T}{T}} \right)
\]
\[
\subset \left\{ |W_T (\hat{\mu}_T - \frac{\gamma}{\theta})| \geq \delta \hat{\sigma}_T^2 \sqrt{T \lambda_T} \right\} \cup \left\{ |2\theta \hat{\sigma}_T^2 - 1| \left| \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \delta \hat{\sigma}_T^2 \sqrt{T \lambda_T} \right\}
\]
\[
\subset \left\{ |W_T (\hat{\mu}_T - \frac{\gamma}{\theta})| \geq \delta E_{\theta, \gamma, x} (\hat{\sigma}_T^2) \sqrt{T \lambda_T} \right\} \cup \left\{ |\hat{\sigma}_T^2 - E_{\theta, \gamma, x} (\hat{\sigma}_T^2)| \geq E_{\theta, \gamma, x} (\hat{\sigma}_T^2) \right\} / 2 \}
\]
\[
\cup \left\{ |2\theta \hat{\sigma}_T^2 - 1| \left| \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right| \geq \delta E_{\theta, \gamma, x} (\hat{\sigma}_T^2) \sqrt{T \lambda_T} \right\} / 4 \}
\]
and
\[
\left( |\hat{\gamma}_T - \gamma| - \frac{2\gamma}{T} \right) \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \geq \delta \sqrt{\frac{\lambda_T}{T}}
\]
\(\sim \begin{cases} 
|\bar{\mu}_T| |W_T(\hat{\mu}_T - \frac{\gamma}{\theta})| / \geq \delta \sigma_T^2 \sqrt{T \lambda_T} / 2 \end{cases} \cup \begin{cases} |2\gamma T - \hat{\mu}_T| \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \geq \delta \sigma_T^2 \sqrt{T \lambda_T} / 2 \end{cases}
\]
\(\cup \begin{cases} 3\frac{\gamma}{2\theta} |W_T(\hat{\mu}_T - \frac{\gamma}{\theta})| \geq \delta E_{\theta,\gamma,x}(\hat{\sigma}_T^2) \sqrt{T \lambda_T} / 4 \end{cases}
\]
\(\cup \begin{cases} \left( 2\gamma T - \hat{\mu}_T - \frac{\gamma}{\theta} \right) \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \geq \delta E_{\theta,\gamma,x}(\hat{\sigma}_T^2) \sqrt{T \lambda_T} / 4 \end{cases}
\).

Therefore, by Lemmas 2.3 and (1), we get the conclusions.

\[\Box\]

**Lemma 3.2.** For each \(\beta, \kappa \in \mathbb{R}\) fixed, \(\left\{ P_{\theta,\gamma,x} \left( \frac{n}{\sqrt{T \lambda_T}} \int_0^T (X_t - \beta) dW_t \in \cdot \right), T \geq 1 \right\} \) satisfies the LDP with speed \(\lambda_T\) and rate function \(J(u) = \frac{\kappa^2 u^2}{\theta^2 + 2(\gamma - \theta)^2}\).

**Proof.** By (2.12) and Lemma 2.3, we can get for any \(\delta > 0\),
\[
\lim_{T \to \infty} \frac{1}{T} \log P_{\theta,\gamma,x} \left( \left| \frac{1}{T} \int_0^T (X_t - \beta)^2 dt - \left( \frac{1}{2\theta} + \frac{1}{\theta^2}(\gamma - \theta)^2 \right) \right| \geq \delta \right) < 0. \tag{3.3}
\]

Therefore, Proposition 1 in [4] yields the conclusion of the lemma.

\[\Box\]

**Proof of Theorem 1.2**

By Lemma 3.1 \(\left\{ P_{\theta,\gamma,x} \left( \frac{T}{\lambda_T} (\hat{\gamma}_T - \theta) \in \cdot \right), T \geq 1 \right\} \) and \(\left\{ P_{\theta,\gamma,x} \left( \frac{T}{\lambda_T} (\hat{\gamma}_T - \gamma) \in \cdot \right), T \geq 1 \right\} \) are exponential equivalent to
\[
\left\{ P_{\theta,\gamma,x} \left( \frac{T}{\lambda_T} \left( X_t - \frac{\gamma}{\theta} \right) dW_t \in \cdot \right), T \geq 1 \right\}
\]

and
\[
\left\{ P_{\theta,\gamma,x} \left( \frac{T}{\lambda_T} \left( \frac{W_T}{T} + \frac{2\gamma}{T} \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t \right) \in \cdot \right), T \geq 1 \right\},
\]

respectively. Noting for \(\gamma \neq 0\), \(\frac{W_T}{T} + \frac{2\gamma}{T} \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t = \frac{2\gamma}{T} \int_0^T \left( X_t - \frac{\gamma}{\theta} \right) dW_t + \frac{1}{2T} dW_L\), Theorem 1.2 follows from Lemma 3.2.
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References


