ON FRACTIONAL STABLE FIELDS INDEXED BY METRIC SPACES

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Abstract
We define and build $H$-fractional $\alpha$-stable fields indexed by a metric space $(E, d)$. We mainly apply these results to spheres, hyperbolic spaces and real trees.

1 Introduction

The $H$-Fractional Brownian Motion $B_H$ [8, 18], indexed by the Euclidean space $(\mathbb{R}^n, ||.|||)$, is a centered Gaussian field such that the variance of its increments is equal to a fractional power of the norm:

$$\mathbb{E}(B_H(M) - B_H(N))^2 = ||MN||^{2H} \quad \forall M, N \in \mathbb{R}^n.$$

In other words, the normalized increments of the $H$-Fractional Brownian are constant in distribution:

$$\frac{B_H(M) - B_H(N)}{||MN||^H} \overset{d}{=} Z \quad \forall M, N \in \mathbb{R}^n,$$

where $Z$ is a centered Gaussian random variable with variance 1. It is well-known that the $H$-Fractional Brownian Motion $B_H$ exists iff $0 < H \leq 1$. [5] proposes to build Fractional Brownian Motions indexed by a metric space $(E, d)$ as centered Gaussian fields which normalized increments are constant in distribution:

$$\frac{B_H(M) - B_H(N)}{d^H(M, N)} \overset{d}{=} Z \quad \forall M, N \in E,$$

where $Z$ is still a centered Gaussian random variable with variance 1. When $(E, d)$ is the sphere or the hyperbolic space endowed with their geodesic distances, [5] proves that the Fractional Brownian Motion exists iff $0 < H \leq 1/2$. When $(E, d)$ is a real tree with its natural distance, [5] proves that the Fractional Brownian Motion exists at least for $0 < H \leq 1/2$.

The following question then arises: what happens when we move from the Gaussian case to the stable case? One knows that there exists several $H$-self-similar $\alpha$-stable fields with stationary
increments indexed by \((\mathbb{R}^n,||.||)\), with various conditions on the fractional index \(H\), providing \(0 < H \leq 1/\alpha\) if \(0 < \alpha \leq 1\) and \(0 < H < 1\) if \(1 < \alpha < 2\) [10]. Let us mention some of them (cf. [13]):

- Linear Fractional Stable Motions: \(0 < H < 1\) and \(H \neq 1/\alpha\),
- \(\alpha\)-Stable Lévy Motions: \(H = 1/\alpha\),
- Log-fractional Stable Motions: \(H = 1/\alpha\),
- Real Harmonizable Fractional Stable Motions: \(0 < H < 1\),
- Lévy-Chentsov fields: \(H = 1/\alpha\),
- \(\beta\)-Takenaka fields: \(0 < \beta < 1\) and \(H = \beta/\alpha\).

All these fields have normalized increments that are constant in distribution:

\[
\frac{X(M) - X(N)}{||MN||^H} \overset{\mathcal{D}}{=} S_\alpha \quad \forall M, N \in \mathbb{R}^n,
\]

where \(S_\alpha\) is a standard symmetric \(\alpha\)-stable random variable, i.e. a random variable which characteristic function is given by:

\[
E(e^{i\lambda S_\alpha}) = e^{-|\lambda|^\alpha}.
\]

We propose to call \(H\)-fractional \(\alpha\)-stable field an \(\alpha\)-stable field \(X(M), M \in E\) which normalized increments are constant in distribution:

\[
\frac{X(M) - X(N)}{d^H(M, N)} \overset{\mathcal{D}}{=} S_\alpha \quad \forall M, N \in E,
\]

where \(S_\alpha\) is a standard symmetric \(\alpha\)-stable random variable.

Let us summarize our main results.

- Non-existence.
  Let \(\beta_E = \sup\{\beta > 0 \text{ such that } d^\beta \text{ is of negative type}\}\). For instance, \(\beta_E\) is equal to 1 for spheres and hyperbolic spaces. We prove that there is no \(H\)-fractional \(\alpha\)-stable field when \(\alpha H > \beta_E\).

- Existence.
  We mainly prove the following. Assume that \(E\) contains a dense countable subset and that \(d\) is a measure definite kernel:

  - if \(0 < \alpha \leq 1\), we construct \(H\)-fractional \(\alpha\)-stable fields for any \(0 < H \leq 1/\alpha\).
  - if \(1 < \alpha < 2\), we construct \(H\)-fractional \(\alpha\)-stable fields for any \(H \in (0, 1/(2\alpha)] \cup [1/2, 1/\alpha]\).
2 Non-existence

Let us first recall the definitions of functions of positive or negative type. Let $X$ be a set.

- A symmetric function $(x, y) \mapsto \phi(x, y), X \times X \to \mathbb{R}^+$ is of positive type if, $\forall x_1, \ldots, x_n \in X, \forall \lambda_1, \ldots, \lambda_n \in \mathbb{R}$
  \[ \sum_{i,j=1}^{n} \lambda_i \lambda_j \phi(x_i, x_j) \geq 0. \]

- A symmetric function $(x, y) \mapsto \psi(x, y), X \times X \to \mathbb{R}^+$ is of negative type if
  - $\forall x \in X, \psi(x, x) = 0$
  - $\forall x_1, \ldots, x_n \in X, \forall \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i = 0$
  \[ \sum_{i,j=1}^{n} \lambda_i \lambda_j \psi(x_i, x_j) \leq 0. \]

Schoenberg’s Theorem [14] implies the equivalence between

- Function $\psi$ is of negative type.
- $\forall x \in X, \psi(x, x) = 0$ and $\forall t \geq 0$, function $\exp(-t\psi)$ is of positive type.

**Lemma 2.1**

*Let $\psi$ be a function of negative type and let $0 < \beta \leq 1$. Then $\psi^\beta$ is of negative type.*

**Proof.**

For $x \geq 0$, and $0 < \beta < 1$, by performing the change of variable $y = \lambda x$, one has:

\[
x^\beta = C_\beta \int_0^{+\infty} e^{-\lambda x} \frac{1}{\lambda^{1+\beta}} d\lambda,
\]

with

\[
C_\beta = \left( \int_0^{+\infty} \frac{e^{-\lambda} - 1}{\lambda^{1+\beta}} d\lambda \right)^{-1}.
\]

Let $\lambda_1, \ldots, \lambda_n$ such that $\sum_{i=1}^{n} \lambda_i = 0$:

\[
\sum_{i,j=1}^{n} \lambda_i \lambda_j \psi^\beta(x_i, x_j) = C_\beta \int_0^{+\infty} \frac{\sum_{i,j=1}^{n} \lambda_i \lambda_j e^{-\lambda \psi(x_i, x_j)}}{\lambda^{1+\beta}} d\lambda.
\]

By Schoenberg’s Theorem, $\sum_{i,j=1}^{n} \lambda_i \lambda_j e^{-\lambda \psi(x_i, x_j)} \geq 0$. Since $C_\beta \leq 0$:

\[
\sum_{i,j=1}^{n} \lambda_i \lambda_j \psi^\beta(x_i, x_j) \leq 0,
\]
and Lemma 2.1 is proved. □

For the metric space \((E, d)\), let us define:

\[
\beta_E = \sup \{ \beta > 0 \text{ such that } d^\beta \text{ is of negative type} \},
\]

with the convention \(\beta_E = 0\) if \(d^\beta\) is never of negative type. Let us note that if \(E\) contains three points \(M_1, M_2, M_3\) such that:

\[
d(M_1, M_2) > d(M_1, M_3),
\]

\[
d(M_1, M_2) > d(M_2, M_3),
\]

then \(\beta_E < \infty\). Indeed, with \(\lambda_1 = -1/2, \lambda_2 = -1/2, \lambda_3 = 1\), one has

\[
\sum_{i,j=1}^{3} \lambda_i \lambda_j d^\beta(M_i, M_j) \sim 1/2 d^\beta(M_1, M_2) > 0 \text{ as } \beta \to +\infty.
\]

**Corollary 2.1**

If \(\beta_E \neq 0\), then \(\{ \beta > 0 \text{ such that } d^\beta \text{ of negative type} \} = (0, \beta_E]\).

**Proof.**

It follows from Lemma 2.1 that \(d^\beta\) is of negative type for \(\beta < \beta_E\), and is never of negative type for \(\beta > \beta_E\). Let \((\beta_p)_{p \geq 0}\) be an increasing sequence converging to \(\beta_E\) (when \(\beta_E < \infty\)).

For all \(M_1, \ldots, M_n \in E\) and \(\lambda_1, \ldots, \lambda_n \in \mathbb{R}\) such that \(\sum_{i=1}^{n} \lambda_i = 0\):

\[
\sum_{i,j=1}^{n} \lambda_i \lambda_j d^\beta_p(M_i, M_j) \leq 0.
\]

Let now perform \(\beta_p \to \beta_E\) in (2):

\[
\sum_{i,j=1}^{n} \lambda_i \lambda_j d^{\beta_E}(M_i, M_j) \leq 0.
\]

It follows that \(d^{\beta_E}\) is of negative type. □

Let us now give some values of \(\beta_E\).

- **Euclidean space \((\mathbb{R}^n, \|\cdot\|)\).**

  One easily checks that function \((x, y) \mapsto \|x - y\|^2\) is of negative type. Indeed, take

  \[
  \lambda_1, \ldots, \lambda_p \text{ with } \sum_{i=1}^{p} \lambda_i = 0 \text{ and } x_1, \ldots, x_p \in \mathbb{R}^n:
  \]

  \[
  \sum_{i,j=1}^{p} \lambda_i \lambda_j \|x_i - x_j\|^2 = -2\|\sum_{i=1}^{p} \lambda_i x_i\|^2 \leq 0.
  \]
Then, by Lemma 2.1, function \((x, y) \mapsto ||x - y||^\beta\) is of negative type when \(0 < \beta \leq 2\). Consider now four vertices \(M_1, M_2, M_3, M_4\) of a square with side length 1 and take \(\lambda_1 = \lambda_3 = 1\) and \(\lambda_2 = \lambda_4 = -1\). Then \(\sum_{i,j=1}^4 \lambda_i \lambda_j ||M_iM_j||^\beta = -8 + 4\sqrt{2}^\beta\) and is strictly positive when \(\beta > 2\). It follows that \(\beta_{R^2} = 2\).

- Space \((\mathbb{R}^n, ||.||_\ell^p)\) where \(||x||_\ell^p = \sum_{i=1}^n |x_i|^\beta\). When \(n \geq 3, q > 2, [6, 7]\) imply \(\beta_{R^n} = 0\).
- Spheres \(S_n = \{ x \in \mathbb{R}^{n+1}, ||x|| = 1\}\) with its geodesic distance. It follows from [5] that \(\beta_{S_n} = 1\).
- Hyperbolic spaces \(H_n = \{ x \in \mathbb{R}^{n+1}, \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1, x_{n+1} \geq 1\}\) with its geodesic distance \(d\). It has been proved by [4] that \(d\) is of negative type. [4, Prop. 7.6] implies that \(\beta_{H_n} = 1\).
- Real trees. A metric space \((T, d)\) is a real tree (e.g. [3]) if the following two properties hold for every \(x, y \in T\):
  - There is a unique isometric map \(f_{x,y}\) from \([0,d(x, y)]\) into \(T\) such that \(f_{x,y}(0) = x\) and \(f_{x,y}(d(x, y)) = y\).
  - If \(\phi\) is a continuous injective map from \([0, 1]\) into \(T\), such that \(\phi(0) = x\) and \(\phi(1) = y\), we have \(\phi([0, 1]) = f_{x,y}([0, d(x, y)]\)).

It has been proved by [17] that the distance on real trees is of negative type: \(\beta_T \geq 1\). One can build trees with \(\beta_T > 1\). Nevertheless, we give a family of simple trees \((T_p)_{p \geq 1}\) such that \(\lim_{p \to +\infty} \beta_{T_p} = 1\). \(A_0\) is the root of the tree. \(A_0\) has \(p\) sons \(A_1, \ldots, A_p\), with:

\[
\begin{align*}
  d(A_0, A_i) &= 1 & i \neq 0, \\
  d(A_i, A_j) &= 2 & i \neq j, i, j \neq 0.
\end{align*}
\]

Choose \(\lambda_0 = 1\) and \(\lambda_i = -1/p\) for \(i = 1, \ldots, p\). Then

\[
\sum_{i,j=0}^p \lambda_i \lambda_j d^\beta(A_i, A_j) = -2 + 2^\beta p - 1.
\]

\(-2 + 2^\beta p - 1/p\) is positive for \(\beta \geq 1 + \log_2 \left(\frac{p}{p-1}\right)\). It follows that \(\beta_{T_p} \leq 1 + \log_2 \left(\frac{p}{p-1}\right)\).

**Proposition 2.1**

*There is no \(H\)-fractional \(\alpha\)-stable fields when \(\alpha H > \beta_E\).*

**Proof.**

We prove Proposition 2.1 by contradiction. Let \(\lambda, \lambda_1, \ldots, \lambda_n \in \mathbb{R}\) and \(M_1, \ldots, M_n \in E\). On one hand:

\[
\sum_{i,j=1}^n \lambda_i \lambda_j \mathbb{E} [\exp(i\lambda(X(M_i) - X(M_j)))] = \mathbb{E} \sum_{i=1}^n \lambda_i \exp(i\lambda X(M_i)) \geq 0.
\]
On the other hand:
\[ \sum_{i,j=1}^{n} \lambda_i \lambda_j \mathbb{E}[\exp(i\lambda(X(M_i) - X(M_j)))] = \sum_{i,j=1}^{n} \lambda_i \lambda_j \exp(-|\lambda|^\alpha d^H(M_i, M_j)). \]

If \( \alpha H > \beta E \), Schoenberg’s Theorem implies that there exists \( \lambda \) such that \( \exp(-|\lambda|^\alpha d^H(M, N)) \) is not of positive type and Proposition 2.1 is proved. \( \square \)

## 3 Construction of \( H \)-fractional \( \alpha \)-stable fields

### 3.1 Main result
Let us recall the definition of a measure definite kernel (cf. [12]).

**Definition 3.1 Measure definite kernel.**
A function \((M,N) \mapsto \psi(M,N)\), from \( E \times E \) onto \( \mathbb{R}^+ \), is a measure definite kernel if there exists a measure space \((H,\sigma(H),\mu)\) and a map \( M \mapsto H_M \) from \( E \) onto \( \sigma(H) \) such that:
\[ \psi(M,N) = \mu(H_M \Delta H_N), \]
where \( \Delta \) denotes the symmetric difference of sets.

For \( \beta > 0, f \in L^\beta(H,\mu) \), define the pseudo-norm:
\[ \| f \|_\beta = \left( \int_H |f|^\beta d\mu \right)^{1/\beta}. \]

It follows that:
\[ \psi(M,N) = \int_H |1_{H_M} - 1_{H_N}|d\mu = \|1_{H_M} - 1_{H_N}\|_\beta^\beta. \]

**Theorem 3.1**
Let \( 1/2 \leq H \leq 1/\alpha \). The following formula, with \( n \geq 1, \lambda_1, \ldots, \lambda_n \in \mathbb{R}, \ M_1, \ldots, M_n \in E, \)
\[ \mathbb{E}\left( \exp\left(i \sum_{j=1}^{n} \lambda_j X(M_j)\right) \right) = \exp\left(-\left\| \sum_{j=1}^{n} \lambda_j 1_{H_{M_j}} \right\|_{1/H}^\alpha \right) \]  
(3)

defines the distribution of an \( \alpha \)-stable field \( X(M), M \in E \) satisfying:
\[ \frac{X(M) - X(N)}{\psi^H(M,N)} \overset{\text{d}}{=} S_\alpha \ \forall M,N \in E, \]  
(4)
where \( S_\alpha \) is a standard symmetric \( \alpha \)-stable random variable.

**Proof.**
We follow Theorem 1 and Lemma 4 of [1]. We have seen that function \((x,y) \mapsto |x-y|\gamma, x,y \in \mathbb{R}\) is of negative type if \( 0 < \gamma \leq 2 \). It follows that function \((f,g) \mapsto |f-g|_1^1/H, f,g \in L^{1/H}(H,\mu)\)
is of negative type when $H \geq 1/2$. Since $\alpha H \leq 1$, one can apply Lemma 2.1: function $(f,g) \mapsto ||f-g||_1^\alpha$, $f,g \in L^{1/H}(\mathcal{H}, \mu)$ is of negative type. Schoenberg’s Theorem implies that, for all $\lambda \in \mathbb{R}$, function $(f,g) \mapsto \exp(-|\lambda|^\alpha ||f-g||_1^\alpha)$, $f,g \in L^{1/H}(\mathcal{H}, \mu)$ is of positive type. (3) is therefore a characteristic function.

Fix now $1 \leq j_0 \leq n$ in (3). We clearly have:

$$
\lim_{\lambda_{j_0} \to 0} \mathbb{E} \left( \exp \left( i \sum_{j=1, j \neq j_0}^n \lambda_j X(M_j) \right) \right) = \exp \left( - \left| \sum_{j=1, j \neq j_0}^n \lambda_j \mathbf{1}_{H_{M_j}} \right|_1^\alpha \right) = \mathbb{E} \left( \exp \left( i \sum_{j=1, j \neq j_0}^n \lambda_j X(M_j) \right) \right).
$$

The Kolmogorov consistency theorem then proves that (3) defines the distribution of an $\alpha$-stable stochastic fields.

Choosing $n = 2$ and $\lambda_1 = -\lambda_2$ in (3) leads to (4).

\begin{proof}

Remark 3.1
One should wonder if function $(f,g) \mapsto ||f-g||_1^\alpha$ is of negative type for $H < 1/2$. Assume that we can choose three disjoints sets $A_1, A_2$ and $A_3$ such that $\mu(A_1) = \mu(A_2) = \mu(A_3) = c > 0$. Put $f = \sum_{i=1}^3 \lambda_i \mathbf{1}_{A_i}$. Then:

$$
||f||_1^\alpha = c^{\alpha H} \left( \sum_{i=1}^3 |\lambda_i|^{1/H} \right)^\alpha H.
$$

But one knows [6, 7] that function $(x,y) \mapsto ||x-y||_1^p$, $x,y \in \mathbb{R}^n$ is never of negative type when $n \geq 3$, $0 < p \leq 2$ and $q > 2$. Function $(f,g) \mapsto ||f-g||_1^\alpha$ is not of negative type for $H < 1/2$.

3.2 Direct applications of Theorem 3.1

3.2.1 Euclidean spaces $(\mathbb{R}^n, ||\cdot||)$, $n \geq 1$

Although it is not our goal, we have a look to the Euclidean spaces. One knows that, for $0 < \beta \leq 1$, functions $(x,y) \mapsto ||x-y||_\ell^\beta$, $x,y \in \mathbb{R}^n$ are measure definite kernels. This is known as Chentsov’s construction ($\beta = 1$) [2] and Takenaka’s construction [15] ($0 < \beta < 1$), see [13, p. 400-402] for a general presentation. Let us briefly describe these two constructions.

- Chentsov’s construction ($\beta = 1$).

For any hyperplane $h$ of $\mathbb{R}^n$, let $r$ be the distance of $h$ to the origin of $\mathbb{R}^n$ and let $s \in S_{n-1}$ be the unit vector orthogonal to $h$. The hyperplane $h$ is parametrized by the pair $(s,r)$. Let $\mathcal{H}$ be the set of all hyperplanes that do not contain the origin. Let $\sigma(\mathcal{H})$ be the Borel $\sigma$-field. Let $\mu(ds, dr) = dsdr$, where $ds$ denotes the uniform measure on $S_{n-1}$ and $dr$ the Lebesgue measure on $\mathbb{R}$. Let $H_M$ be the set of all hyperplanes separating the origin and the point $M$. Then, there exists a constant $c > 0$ such that

$$
||MN|| = c\mu(H_M \Delta H_N).
$$
• Takenaka’s construction \((0 < \beta < 1)\).

A hypersphere in \(\mathbb{R}^n\) is parametrized by a pair \((x, \lambda)\), where \(x \in \mathbb{R}^n\) is its center and \(\lambda \in \mathbb{R}^+\) its radius. Let \(H\) be the set of all hyperspheres in \(\mathbb{R}^n\). Let \(\sigma(H)\) be the Borel \(\sigma\)-field. \(\mu_\beta\) is the measure \(\mu_\beta(dx, d\lambda) = \lambda^{\beta-n-1}dxd\lambda\). Let \(H_M\) be the set of hyperspheres separating the origin and the point \(M\). Then, there exists a constant \(c_\beta > 0\) such that

\[ ||MN||^\beta = c_\beta \mu_\beta(H_M \Delta H_N). \]

Theorem 3.1 can therefore be applied with \(\psi = ||.||^\beta\), \(0 < \beta \leq 1\). This leads \(H\)-fractional \(\alpha\)-stable fields for any \(0 < \beta \leq 1\) and any \(H\) providing \(1/2 < H \leq 1/\alpha\). The range of feasible parameters is therefore \(0 < H \leq 1/\alpha\).

### 3.2.2 Spheres \(S_n\)

When [9] introduces the Spherical Brownian Motion, he proves that the geodesic distance \(d\) is a measure definite kernel. Indeed, for any point \(M\) on the unit sphere, define a half-sphere by:

\[ H_M = \{N \in S_n, d(M, N) \leq \pi/2\}. \]

Let \(ds\) be the uniform measure on \(S_n\), let \(\omega_n\) be the surface of the sphere, and define the measure \(\mu\) by:

\[ \mu(ds) = \frac{\pi}{\omega_n}ds. \]

Then:

\[ d(M, N) = \mu(H_M \Delta H_N). \]

Theorem 3.1 can be applied with \(\psi = d\). This leads to \(H\)-fractional \(\alpha\)-stable fields for any \(H\) providing \(1/2 \leq H \leq 1/\alpha\).

### 3.2.3 Hyperbolic spaces \(H_n\)

The geodesic distance is a measure definite kernel [16, 11]. The proof is more technical and we give only a rough outline. \(H_n\) is considered as a subset of the real projective space \(P^n(\mathbb{R})\).

Let \(H_M\) be the set of hyperplanes that separates \(M\) and the origin 0 of \(H_n\) in \(P^n(\mathbb{R}) - l_\infty\).

Let \(\mu\) be a measure on \(H_n\) invariant under the action of the Lorentz group. Then, up to a normalizing constant, the geodesic distance \(d\) can be written as:

\[ d(M, N) = \mu(H_M \Delta H_N). \]

Theorem 3.1 can be applied with \(\psi = d\). This leads to \(H\)-fractional \(\alpha\)-stable fields for any \(H\) providing \(1/2 \leq H \leq 1/\alpha\).

### 3.2.4 Real trees

Let us shortly explain the construction given in [17]. Fix \(O\) in the tree \(T\). Set \(H_M\) be the geodesic path between \(O\) and \(M\). Then \(d(M, N) = \mu(H_M \Delta H_N)\) where \(\mu\) is the Valette’s measure of the tree: distance \(d\) is a measure definite kernel.

Theorem 3.1 can be applied with \(\psi = d\). This leads to \(H\)-fractional \(\alpha\)-stable fields for any \(H\) providing \(1/2 \leq H \leq 1/\alpha\).
4 Space with countable dense subspace

We will now extend the result of Theorem 3.1.
Assume that \( E \) contains a countable dense subset \( \Gamma \) and that distance \( d \) is a measure definite kernel. A measure definite kernel is always of negative type [12, Prop. 1.1]. It follows from Lemma 2.1 that \( (M,N) \mapsto d^\beta(M,N), M,N \in E, 0 < \beta \leq 1 \), is of negative type. [12, Prop. 1.4] proves that the square root of a function of negative type defined on a countable space is a measure definite kernel. It follows that \( (M,N) \mapsto d^\beta/2(M,N), M,N \in \Gamma, 0 < \beta \leq 1 \), is a measure definite kernel since \( \Gamma \) is countable. Theorem 3.1 can be applied with \( \psi(M,N) = d^\beta/2(M,N), M,N \in \Gamma, 0 < \beta \leq 1 \): we have build a field \( X(M), M \in E \), \( \alpha \)-stable, it has finite moments of order \( 0 < \alpha' < \alpha \) and there exists a constant \( c > 0 \) (cf. [13, Prop. 1.2.17]) such that, for all \( N,N' \in \Gamma \):

\[
\mathbb{E}|X(N) - X(N')|^{\alpha'} = cd^{\alpha'/2}(N,N').
\]

Let \( M \in E - \Gamma \) and let \( N \to M, N \in \Gamma \). From (5), one can define \( X(M) \) as:

\[
X(M) \overset{p}{=} \lim_{N \to M, N \in \Gamma} X(N).
\]

We have therefore build an \( \alpha \)-stable field \( X(M), M \in E \) satisfying:

\[
\frac{X(M) - X(N)}{d^{\beta/2}(M,N)} \overset{D}{=} S_\alpha \forall M,N \in E,
\]

where \( S_\alpha \) is a standard symmetric \( \alpha \)-stable random variable.

Let us now apply this construction to the spheres and hyperbolic spaces with their geodesic distances. We build \( \beta H/2 \)-fractional \( \alpha \)-stable fields with any \( 0 < \beta \leq 1, 1/2 \leq H \leq 1/\alpha \).

Let us summarize this construction with the previous construction of sections 3.2.2 and 3.2.3. We are able to build \( H \)-fractional \( \alpha \)-stable fields in the following cases:

- when \( \alpha \leq 1 \), with any \( 0 < H \leq 1/\alpha \); and one knows from Proposition 2.1 that \( H > 1/\alpha \) is forbidden;

- when \( 1 < \alpha < 2 \), with any \( 0 < H \leq 1/(2\alpha) \) and \( 1/2 \leq H \leq 1/\alpha \); and one knows from Proposition 2.1 that \( H > 1/\alpha \) is still forbidden; the interval \((1/(2\alpha), 1/2)\) is “missing”.

Remark 4.1
One doesn’t know if \( d^\gamma \) is a measure definite kernel for \( 1/2 < \gamma < 1 \). This is the reason of the “missing” interval \((1/(2\alpha), 1/2)\).

References


