ON UNIQUENESS OF A SOLUTION OF $Lu = u^\alpha$ WITH GIVEN TRACE

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Abstract

A boundary trace $(\Gamma, \nu)$ of a solution of $\Delta u = u^\alpha$ in a bounded smooth domain in $\mathbb{R}^d$ was first constructed by Le Gall [12] who described all possible traces for $\alpha = 2, d = 2$ in which case a solution is defined uniquely by its trace. In a number of publications, Marcus, Véron, Dynkin and Kuznetsov gave analytic and probabilistic generalization of the concept of trace to the case of arbitrary $\alpha > 1, d \geq 1$. However, it was shown by Le Gall [13] that the trace, in general, does not define a solution uniquely in case $d \geq (\alpha + 1)/(\alpha - 1)$. He offered a sufficient condition for the uniqueness and conjectured that a uniqueness should be valid if the singular part $\Gamma$ of the trace coincides with the set of all explosion points of the measure $\nu$. Here, we establish a necessary condition for the uniqueness which implies a negative answer to the above conjecture.

1 Introduction and Results

1.1 Moderate solutions

Let $L$ be a second order uniformly elliptic differential operator with smooth coefficients in $\mathbb{R}^d$ and let $E \subset \mathbb{R}^d$ be a bounded smooth domain. We consider a class $U$ of all positive solutions of the equation

$$Lu = u^\alpha \quad \text{in } E$$

(1.1)

where $\alpha \in (1, 2]$ is a parameter. A solution $u$ is called moderate if $u \leq h$ for an $L$-harmonic function $h$. The class of all moderate solutions is denoted by $U_1$.

For every moderate solution $u$, there exists a minimal $L$-harmonic function that dominates $u$. It is called the minimal (L-harmonic) majorant of $u$. A solution $u$ can be recovered from...
its majorant as the maximal solution to (1.1) dominated by \( h \). Moreover, \( u \) is related to its minimal majorant \( h \) by the integral equation

\[
u(x) + \Pi_x \int_0^\zeta u^\alpha(\xi_s) \, ds = h(x).
\]  

(1.2)

Here \((\xi_\zeta, \Pi_x)\) is the corresponding \( L \)-diffusion in \( E \) and \( \zeta \) is its life time. See [5] for more detail. Every positive \( L \)-harmonic function \( h \) has a unique representation

\[
(h(x) = \int_{\partial E} k(x, y) \nu(dy)
\]

(1.3)

where \( k(x, y) \) is the Poisson kernel for \( L \) in \( E \) and \( \nu \) is a finite measure on \( \partial E \). We denote by \( h_\nu \) the function given by (1.3). For a moderate solution \( u \in \mathcal{U}_1 \), we write \( u = u_\nu \) if \( h_\nu \) is the minimal majorant of \( u \).

### 1.2 Superdiffusions and stochastic boundary values

An \((L, \alpha)\)-superdiffusion is a probabilistic model for an evolution of a random cloud of branching particles. A spatial movement of particles is described by an \( L \)-diffusion, and \( \alpha \in (1, 2] \) characterizes branching. See, for instance, [2]. To every open set \( D \) there corresponds a random measure \( (X_D, P_\mu) \) on \( \partial D \), called the exit measure from \( D \). It represents the total accumulation of mass on \( \partial D \) assuming that the evolution starts from \( \mu \) and particles are instantly frozen if they reach the complement of \( D \). Relations between \( X_D \) and equation (1.1) can be described as follows. Let \( f \) be a positive continuous function on \( \partial E \). The function

\[
u(x) = -\log P_x e^{-(f; X_E)}
\]

(1.4)

where \( P_x \) stands for \( P_{\beta x} \), is the only solution of the boundary value problem

\[
Lu = u^\alpha \quad \text{in } E,
\]

\[
u = f \quad \text{on } \partial E.
\]

(1.5)

An arbitrary solution \( u \) of (1.1) can also be represented in a form similar to (1.4) in terms of its stochastic boundary value \( Z_u \) (cf. [3]). It can be defined as a limit

\[
Z_u = \lim\langle u, X_{D_n} \rangle
\]

(1.6)

where \( D_n \) is an increasing sequence of bounded smooth domains approximating \( E \). A solution \( u \) can be recovered from its stochastic boundary value by the formula

\[
u(x) = -\log P_x e^{-Z_u}.
\]

(1.7)

We write \( Z_\nu \) instead of \( Z_u \). See [3] for more detail.

We define the range \( R \) of a superdiffusion in \( E \) as the minimal closed set that supports all \( X_D \) for \( D \subset E \). A set \( \Gamma \subset \partial E \) is called a polar set for the superdiffusion if, for any \( x \),

\[
P_x \{ R \cap \Gamma \neq \emptyset \} = 0.
\]

According to [6], the class of polar sets coincides with the class of all removable boundary singularities for the equation (1.1). By [4], the equation (1.2) has a solution if and only if the corresponding measure \( \nu \) does not charge polar sets. Therefore the mapping \( \nu \rightarrow u_\nu \) defines a 1-1 correspondence between the class \( N_1 \) of all finite measures on
\( \partial E \) which don’t charge polar sets and the class \( \mathcal{U}_1 \) of all moderate solutions of (1.1); see [5], [4], [9], [7].

For every Borel subset \( B \subset \partial E \),

\[
    w_B(x) = -\log P_x \{ R \cap B = \emptyset \}
\]

(1.8)

is a solution of (1.1). Its stochastic boundary value is given by the formula \( Z_B = Z_{w_B} = \infty \mathbb{1}_{(R \cap B \neq \emptyset)} \). If \( B \) is closed, then \( w_B \) is the maximal solution of (1.1) such that \( w_B = 0 \) on \( \partial E \setminus B \). See [3], Sect. 6.

### 1.3 \( \sigma \)-moderate solutions

A solution \( u \) of (1.1) is called \( \sigma \)-moderate if there exists an increasing sequence of moderate solutions \( u_n \) such that \( u_n \uparrow u \) as \( n \to \infty \). It follows from (1.2) that the corresponding measures \( \nu_n \) also increase to some measure \( \nu \). The measure \( \nu \) does not charge polar sets, but it may be not finite and not even \( \sigma \)-finite. However, it is always \( \Sigma \)-finite. We denote by \( \mathcal{N}_0 \) the class of all \( \Sigma \)-finite measures that don’t charge polar sets. Every measure \( \nu \in \mathcal{N}_0 \) can be represented as a limit of an increasing sequence of finite measures \( \nu_n \) and therefore defines a \( \sigma \)-moderate solution \( u = \lim u_{\nu_n} \). We denote this solution by \( u_{\nu} \) and we write \( Z_{\nu} \) for its stochastic boundary value. (It follows from [9], Theorem 4.2 that \( u_{\nu} \) and \( Z_{\nu} \) do not depend on the choice of \( \nu_n \uparrow \nu \).) Every \( \sigma \)-moderate solution can be represented this way. However, in contrast to moderate solutions, this representation is not unique. \( \sigma \)-moderate solutions have been studied in Section 4 of [9] by means of continuous linear additive functionals.

The class of all \( \sigma \)-moderate solutions is denoted by \( \mathcal{U}_0 \). Existence of non-\( \sigma \)-moderate solutions remains an open question: all known elements of \( \mathcal{U} \) either belong to \( \mathcal{U}_0 \) or, at least, it is not proved that this is not true. See [11], [7].

### 1.4 Sweeping and the trace

First definition of the trace was introduced by Le Gall [12], [14], [13], who used it to describe all solutions of the equation \( \Delta u = u^2 \) in a smooth planar domain. In a more general setting, a definition of a trace was introduced by Marcus and Véron [15], [16], [17], [18] and, in a probabilistic way, by Dynkin and Kuznetsov [9], [8].

Let \( u \in \mathcal{U} \). For a closed set \( B \subset \partial E \), we define \( Q_B(u) \) as the maximal element of \( \mathcal{U} \) such that \( Q_B(u) \leq u \) and \( Q_B(u) = 0 \) on \( \partial E \setminus B \). We consider the maximal open subset \( O \) of \( \partial E \) such that \( Q_B(u) \) is moderate for every compact \( B \subset O \) and we set \( \Gamma = O^c \). It can be shown that there exists a Radon measure \( \nu \) on \( O \) such that \( Q_{\nu}(u) = u_{\nu_B} \) for every compact \( B \subset O \) where \( \nu_B \) stands for the restriction of \( \nu \) to \( B \). The pair \( (\Gamma, \nu) \) is called the trace of \( u \). Cf. [9].

Let \( \nu \) be a measure on \( \partial E \). A point \( x \in \partial E \) is called an explosion point for \( \nu \) if \( \nu(O) = \infty \) for every open set \( O \) containing \( x \). The collection of all explosion points of \( \nu \) is denoted by \( Ex(\nu) \). Clearly, \( Ex(\nu) \) is a closed set. Let \( \Gamma \) be a closed subset of \( \partial E \) and \( \nu \) be a Radon measure on \( \Gamma^c \) not charging polar sets. The pair \( (\Gamma, \nu) \) is called normal if there exists no nontrivial relatively open polar subset \( B \subset \Gamma \setminus Ex(\nu) \).

**Proposition 1.1 (See [9])**. The trace \( (\Gamma, \nu) \) of a solution \( u \in \mathcal{U} \) is always a normal pair. Each normal pair \( (\Gamma, \nu) \) is the trace of some solution \( u \). The maximal solution with the given trace \( (\Gamma, \nu) \) is given by the formula

\[
    w_{\Gamma, \nu}(x) = -\log P_x \{ R \cap \Gamma = \emptyset, e^{-Z_{\nu}} \}.
\]

(1.9)
1.5 Essential explosion points

For an arbitrary Borel set $B \subset \partial E$, put

$$\text{Cap}_R(B) = P_c\{R \cap B \neq \emptyset\} \quad (1.10)$$

where $c$ is a reference point and $R$ is the range of the $(L, \alpha)$-superdiffusion in $E$. According to [1], Theorem III.32, $\text{Cap}_R(B)$ is a Choquet capacity. By [3], Sect. 6.2, $\text{Cap}_R(B) = 0$ if and only if $B$ is polar.

Let $x \in \text{Ex}(\nu)$. We call $x$ a point of non-essential explosion if there exists a neighborhood $U$ of $x$ and a sequence of open sets $O_n \subset U$ such that $\text{Cap}_R(O_n) \downarrow 0$ as $n \to \infty$ and $\nu(U \setminus O_n) < \infty$ for all $n$. Otherwise $x$ is called a point of essential explosion. We denote the set of all essential explosion points by $\text{Ess}(\nu)$. Note that $\text{Ex}(\nu) = 0$ if single points on the boundary are not polar (this happens if $d < (\alpha + 1)/(\alpha - 1)$; see [10], [6]).

Properties of $\text{Ess}(\nu)$ can be summarized as follows.

**Theorem 1.1.** Let $\nu$ be a $\Sigma$-finite measure that doesn’t charge polar sets. Then:

(i) The set $\text{Ess}(\nu)$ is always closed;

(ii) There exist open sets $U_n \supset \text{Ess}(\nu)$ such that $\text{Cap}_R(U_n) \downarrow \text{Cap}_R(\text{Ess}(\nu))$ and $\nu(U_n) < \infty$;

(iii) $\text{Ess}(\nu)$ is either empty or non-polar;

(iv) $\nu$ is $\sigma$-finite on the complement of $\text{Ess}(\nu)$.

The following result clarifies the difference between $\text{Ex}(\nu)$ and $\text{Ess}(\nu)$.

**Theorem 1.2.** Let $d \geq (\alpha + 1)/(\alpha - 1)$. For every closed set $\Gamma \subset \partial E$, there exists a $\sigma$-finite measure $\nu \in \mathcal{N}_0$ such that $\text{Ex}(\nu) = \Gamma$ and $\text{Ess}(\nu)$ is empty. The measure $\nu$ can be chosen to have a density with respect to the surface measure.

Main result of the paper is given by the following

**Theorem 1.3.** Let $d \geq (\alpha + 1)/(\alpha - 1)$ and let $(\Gamma, \nu)$ be a normal pair. If $\Gamma \setminus \text{Ess}(\nu)$ is not polar, then there exist at least two solutions with the trace $(\Gamma, \nu)$.

**Remarks.** 1. For $\nu = 0$ this is, essentially, Proposition 3 in [13] (to be precise, [13] is devoted to the initial trace of the corresponding semilinear parabolic equation. However, the arguments can be easily extended to an elliptic case.)

2. Le Gall proved in [13] that the uniqueness takes place if $\Gamma$ is polar and that there is no uniqueness if $\nu = 0$ and $\Gamma$ is not polar. He also conjectured that the uniqueness is valid if $\Gamma = \text{Ex}(\nu)$. The following example shows that it is not true. Let $\Gamma$ be a non-polar closed subset of $\partial E$ with the surface measure 0 and let $\nu$ be the measure constructed in Theorem 1.2. Since $\nu$ has the density with respect to the surface measure, $\nu(\Gamma) = 0$. By Theorem 1.2, $\Gamma = \text{Ex}(\nu)$ and therefore $(\Gamma, \nu)$ is a normal pair. However, $\text{Ess}(\nu)$ is empty. By Theorem 1.3, there exist at least two solutions with the trace $(\Gamma, \nu)$. This implies a negative answer to the above conjecture.
1.6 Remarks on fine trace

To overcome the difficulties related to the non-uniqueness, Dynkin and Kuznetsov have defined in [11], [7] a fine trace of a solution and they have shown that \( \sigma \)-moderate solutions are defined uniquely by their fine traces. The fine trace is a pair \((\Gamma, \nu)\) where \( \Gamma \) is closed in a certain topology (fine topology) related to the equation (1.1) and \( \nu \) is a \( \sigma \)-finite measure on the complement of \( \Gamma \) not charging polar sets. See [11], [7] for all details.

**Theorem 1.4.** Let \( \nu \) be a \( \Sigma \)-finite measure that doesn’t charge polar sets. If \( \text{Ess}(\nu) \) is empty, then the fine trace of \( u_\nu \) is equal to \((\Gamma, \nu)\) with polar \( \Gamma \).

By applying this result to measures \( \nu \) constructed in Theorem 1.2, we see that the corresponding solutions \( u_\nu \) are not determined by their traces and they can be recovered from their fine traces.

2 Proofs

2.1 Some Lemmas

We start with an important lemma.

**Lemma 2.1.** Let \( u \leq v \) be two solutions of (1.1). If \( u(c) = v(c) \) at some interior point \( c \in E \), then \( u = v \) everywhere in \( E \).

**Proof.** It is sufficient to show that \( u = v \) in any bounded smooth domain \( D \) such that \( c \in D \) and \( D \subset E \). Solutions \( u \) and \( v \) are bounded and continuous in \( D \) and therefore they admit a representation

\[
\begin{align*}
u(x) &= -\log P_x e^{-\langle u, X_D \rangle}, \\
v(x) &= -\log P_x e^{-\langle v, X_D \rangle}.
\end{align*}
\]

Since \( u \leq v \), we conclude from (2.1) that \( \langle u, X_D \rangle = \langle v, X_D \rangle \) \( P \)-a.s. Therefore

\[
\Pi_x u(\xi_{\tau_D}) = P_x \langle u, X_D \rangle = P_x \langle v, X_D \rangle = \Pi_x v(\xi_{\tau_D})
\]

and \( u = v \) on \( \partial D \). By (2.1), this yields \( u = v \) in \( D \). \( \square \)

As a next step, we compute the trace of \( w_B \) for Borel \( B \).

**Lemma 2.2.** Let \( B \) be a Borel subset of \( \partial E \). The trace of \( w_B \) is equal to \((\Gamma, 0)\) where \( \Gamma \) is the smallest closed set such that \( B \setminus \Gamma \) is polar.

**Proof.** Let \( (\Gamma, \nu) \) be the trace of \( w_B \). Suppose \( B \setminus \Gamma \) is not polar. There exists a non-polar compact \( K \subset B \setminus \Gamma \). Since \( K \subset B \), \( w_B \geq w_K \) and therefore the sweeping \( Q_K(w_B) \geq Q_K(w_K) = w_K \). But, \( w_K \) is not moderate. Suppose now that a closed \( \hat{\Gamma} \) is such that \( B \setminus \hat{\Gamma} \) is polar. Then \( w_B \leq w_{\hat{\Gamma}} \) by (1.8) and therefore \( Q_K(w_B) \leq Q_K(w_{\hat{\Gamma}}) \) for every \( K \). This implies \( Q_K(w_B) = 0 \) for every compact \( K \subset \Gamma^c \), and therefore \( \Gamma \subset \hat{\Gamma} \). Same argument applied to \( \Gamma \) instead of \( \hat{\Gamma} \) shows that \( \mu = 0 \). \( \square \)

**Lemma 2.3.** Let \( \nu \in N_0 \). The trace of \( u_\nu \) is equal to \((\Gamma, \mu)\) where \( \Gamma = \text{Ex}(\nu) \) and \( \mu \) coincides with the restriction of \( \nu \) to \( \Gamma^c \).
Proof. Let $B \subset \partial E$ be a compact. Clearly, $u_{\nu} \geq u_{\nu_B}$ where $\nu_B$ is the restriction of $\nu$ to $B$. Therefore

$$Q_B(u_{\nu}) \geq Q_B(u_{\nu_B}) \geq u_{\nu_B}. \tag{2.2}$$

Suppose now that $B \cap \text{Ex}(\nu) = \emptyset$. Show that

$$Q_B(u_{\nu}) = u_{\nu_B} \tag{2.3}$$

for such $B$. There exists a relatively open $U \subset \partial E$ such that $B \subset U$ and $\nu(U) < \infty$. Let $\lambda$ and $\kappa$ be the restrictions of $\nu$ to $U$ and $U^c$, respectively. Let $u_1 = u_{\lambda}$ and $u_2 = u_{\kappa}$. By [3], Theorem 2.3,

$$u \leq u_1 + u_2$$

and therefore

$$Q_B(u) \leq Q_B(u_1) + Q_B(u_2). \tag{2.4}$$

However, solution $u_1$ is moderate and therefore $Q_B(u_1) = u_{\lambda_B} = u_{\nu_B}$. On the other hand, $\kappa$ vanishes on $U$ and therefore $Q_B(u_2) = 0$ by [9], 4.4. A and Theorem 3.1. Combining (2.2) and (2.4), we get (2.3).

The statement of the lemma follows from (2.2), (2.3) and the definition of the trace. \qed

Lemma 2.4. For every Borel set $B \subset \partial E$, and every $\nu \in \mathcal{N}_0$,

$$w_{B,\nu}(x) = -\log P_x\{R \cap B = \emptyset, e^{-Z_r}\}$$

is a solution of (1.1). Its trace $(\Gamma, \mu)$ can be characterized by the following properties. The set $\Gamma$ is the smallest closed set such that $\Gamma \supset \text{Ex}(\nu)$ and $B \setminus \Gamma$ is polar, and $\mu$ is the restriction of $\nu$ to $\Gamma^c$.

Proof. Note that $w_{B,\nu}(x) = -\log P_x e^{-Z_B - Z_r}$ and therefore the first part follows easily from Theorems 2.3 and 6.1 in [3]. Second part follows easily from Lemmas 2.2 and 2.3 and from the inequalities

$$w_B \leq w_{B,\nu}, \quad u_{\nu} \leq w_{B,\nu}, \quad w_{B,\nu} \leq u_{\nu} + w_B$$

(for the last inequality, see, e.g., [3], Theorem 2.3). \qed

Lemma 2.5. Let $\Gamma$ be a closed subset of $\partial E$ and let $B \supset \Gamma$ be such that $B \setminus \Gamma$ is not polar. Then $\text{Cap}_R(B) > \text{Cap}_R(\Gamma)$.

Proof. Suppose $\text{Cap}_R(B) = \text{Cap}_R(\Gamma)$. Then $w_B(x) = w_{\Gamma}(x)$ and therefore $w_B = w_{\Gamma}$ everywhere by Lemma 2.1. By assumption, there exists a compact $K \subset B \setminus \Gamma$ such that $\text{Cap}_R(K) > 0$. Clearly, $w_K \leq w_B$ and therefore $w_K \leq w_{\Gamma}$. However, $w_K = 0$ on $\partial E \setminus K$ and $w_{\Gamma} = 0$ on $\partial E \setminus \Gamma$, which implies $w_K = 0$ on $\partial E$ and therefore $w_K = 0$ in $E$, that is $\text{Cap}_R(K) = 0$. \qed

2.2 Proof of Theorem 1.1

1°. Let $x \notin \text{Ess}(\nu)$. If $x \notin \text{Ex}(\nu)$, then there exists an open set $U \subset \partial E$ such that $x \in U$ and $\nu(U) < \infty$. By definition of explosion points, all points of $U$ do not belong to $\text{Ex}(\nu) \supset \text{Ess}(\nu)$. Suppose now that $x$ is a point of non-essential explosion and $U$ is as in definition. Clearly, any $y \in \text{Ex}(\nu) \cap U$ must also be a point of non-essential explosion. Therefore each point $x \notin \text{Ess}(\nu)$ has a neighborhood that does not intersect with $\text{Ess}(\nu)$. Hence $\text{Ess}(\nu)$ is closed.
On the other hand, $A_1$ is

\[ \text{On the other hand, the set } O \text{ is } \]

Indeed, let $B$ be an open set such that $\text{Cap}_R(B) \leq \text{Cap}_R(\text{Ess}(\nu)) + \varepsilon$. Denote $B = U^c$ and $F = B \cap \text{Ex}(\nu)$. By construction, $F$ consists of non-essential explosion points. For $x \in F$, denote by $U_x$ the neighborhood of $x$ described in the definition of a non-essential explosion point. Open sets $U_x$ cover a compact set $F$ and therefore there exists a finite set $x_1, \ldots, x_k \in F$ such that $F \subset U_{x_1} \cup \cdots \cup U_{x_k}$. For each $x_j$, there exists an open set $O_j \subset U_{x_j}$ such that $\nu(U_{x_j} \setminus O_j)$ is finite and $\text{Cap}_R(O_j) \leq \varepsilon/k$. Put $\tilde{O} = U \cup O_1 \cup \ldots \cup O_k$. By construction,

\[ \text{Cap}_R(O) \leq \text{Cap}_R(U) + \sum_j \text{Cap}_R(O_j) \leq \text{Cap}_R(\text{Ess}(\nu)) + 2\varepsilon \]

On the other hand, the set $O^c$ is contained in the union of the sets $K_j = U_{x_j} \setminus O_j$ and the set $K = E \setminus \{U \cup U_{x_1} \cup \cdots \cup U_{x_k}\}$. Sets $K_j$ have finite measure by construction. The set $K$ is a compact set disjoint from $\text{Ex}(\nu)$ and therefore $\nu(K) < \infty$. (Recall that $\nu$ is a Radon measure on the complement of $\text{Ex}(\nu)$ and therefore $\nu(K) < \infty$ for every compact $K$ that contains no explosion points.)

3°. Suppose $\text{Ess}(\nu)$ is polar. Let $O_n$ be the sequence constructed in (ii). Put $U = E$. Since $\text{Cap}_R(O_n) \downarrow 0$ and $\nu(E \setminus O_n) < \infty$, all explosion points are non-essential.

4°. Again, let $O_n$ be the sequence constructed in (ii). Denote $B = \cap_n O_n$. By construction, $\nu$ is $\sigma$-finite on $B^c$. Besides, $\text{Cap}_R(B) = \text{Cap}_R(\text{Ess}(\nu))$ and therefore $B \setminus \text{Ess}(\nu)$ is polar by Lemma 2.5. Hence $\nu(B \setminus \text{Ess}(\nu)) = 0$ and $\nu$ is $\sigma$-finite on the complement of $\text{Ess}(\nu)$ as well.

\[ \text{2.3 Essential explosion points and stochastic boundary values} \]

\[ \text{Lemma 2.6. Let } \nu \in \mathcal{N}_0. \text{ Then } Z_\nu \text{ is finite a.s. on the set } \{\mathcal{R} \cap \text{Ess}(\nu) = \emptyset\}. \]

\[ \text{Proof. Let } K \text{ be a compact with } \nu(K) < \infty \text{ and let } \nu_K \text{ be the restriction of } \nu \text{ to } K. \text{ The solution } u_{\nu_K} \text{ is moderate, and therefore } Z_{\nu_K} \text{ is finite a.s. As a first step, we prove that} \]

\[ Z_\nu = Z_{\nu_K} \text{ a.s. on the set } \{\mathcal{R} \cap K^c = \emptyset\}. \]

Indeed, let $B \subset K^c$ be a compact. Then $u_{\nu_K} \leq w_B$ and

\[ Z_{\nu_B} \leq Z_{w_B} = \infty 1_{\{\mathcal{R} \cap B \neq \emptyset\}} \]

and therefore $Z_{\nu_B} = 0$ on $\{\mathcal{R} \cap K^c = \emptyset\}$. Let now $B_n$ be an increasing sequence of compacts with the union $K^c$. Since $\nu$ is the increasing limit of $\nu_n = \nu_K + \nu_{B_n}$,

\[ Z_\nu = \lim Z_{\nu_n} = \lim Z_{\nu_K} + Z_{\nu_{B_n}} \]

(see [3], Sect. 3.8) and therefore

\[ Z_\nu = Z_{\nu_K} \text{ on } \{\mathcal{R} \cap K^c = \emptyset\}. \]

By Theorem 1.1(ii), there exists a decreasing sequence of open sets $O_n$ such that $\nu(O_n^c) < \infty$, $\text{Ess}(\nu) \subset O_n$ and

\[ P_\nu \{\mathcal{R} \cap O_n \neq \emptyset\} \downarrow P_\nu \{\mathcal{R} \cap \text{Ess}(\nu) \neq \emptyset\}. \]

(2.6)

Denote by $K_n$ the complement of $O_n$. By (2.5), $Z_\nu$ is a.s. finite on the sets $A_n = \{\mathcal{R} \cap O_n = \emptyset\}$. On the other hand, $A_n \uparrow \{\mathcal{R} \cap \text{Ess}(\nu) = \emptyset\}$ by (2.6).
2.4 Proof of Theorem 1.2

In order to construct the measure \( \nu \), take an arbitrary countable dense subset \{\( x_k \)\} of \( \Gamma \). For every \( k \), let \( \phi_k(x) = |x - x_k|^{-d} \). \( \phi_k \) is a continuous positive function on \( \partial D \setminus \{x_k\} \), which is not integrable over every neighborhood of \( x_k \). The assumption \( d \geq (\alpha + 1)/2 \) implies \( \text{Cap}_R(x_k) = 0 \). Hence \( \lim_{n \to \infty} \text{Cap}_R(\phi_k > n) = 0 \) and therefore \( \text{Cap}_R(\phi_k > C_k) \leq 2^{-k} \) for sufficiently large \( C_k \).

Put \( f = \sup_k \phi_k/C_k \) and \( \nu(dx) = f(x)\sigma(dx) \), where \( \sigma(dx) \) is the surface measure. By increasing \( C_k \), if necessary, we can make \( f \) bounded on a positive distance from \( \Gamma \). For this reason, \( \text{Ex}(\nu) \subset \Gamma \). On the other hand, \( x_n \) are dense in \( \Gamma \). Therefore, if \( x \in \Gamma \) and \( U \) is a neighborhood of \( x \), then \( U \) contains at least one of the points \( x_n \) and therefore \( f \) is not integrable over \( U \).

Hence \( \Gamma = \text{Ex}(\nu) \).

Denote \( O_n = \{f > n\} = \bigcup_k \{\phi_k > nC_k\} \). Since \( \phi_k \) are continuous, \( O_n \) is open for every \( n \).

Besides, \( \nu(O_n^c) \leq n\sigma(O_n^c) < \infty \) for every \( n \). By construction, \( \text{Cap}_R(O_n) \leq \sum \text{Cap}_R(\phi_k > nC_k) \to 0 \) as \( n \to \infty \) by the dominated convergence theorem. Hence, \( \nu \) has no essential explosion points. \( \square \)

2.5 Proof of Theorem 1.3

1°. Suppose \( \Gamma = \text{Ex}(\nu) \) and \( \text{Ex}(\nu) \setminus \text{Ess}(\nu) \) is not polar. By Theorem 1.1(i), both \( \text{Ex}(\nu) \) and \( \text{Ess}(\nu) \) are closed. Therefore Lemma 2.5 implies that \( \text{Cap}_R(\text{Ex}(\nu)) > \text{Cap}_R(\text{Ess}(\nu)) \) and therefore

\[
P_c\{R \cap \text{Ex}(\nu) \neq \emptyset, R \cap \text{Ess}(\nu) = \emptyset\} > 0.
\]

(2.7)

Let \( v = w_{\Gamma,\nu} \) and \( u = u_{\nu} \). By Proposition 1.1, \( v \) is the maximal solution with the trace \((\Gamma, \nu)\). By Lemma 2.3, \( u \) also has the trace \((\Gamma, \nu)\). However,

\[
Z_v \geq Z_\Gamma = \infty 1_{\mathcal{R} \cap \Gamma \neq \emptyset}
\]

and

\[
Z_u < \infty \quad \text{a.s. on } \mathcal{R} \cap \text{Ess}(\nu) = \emptyset
\]

by Lemma 2.6. By (2.7), \( Z_u \neq Z_v \) with positive probability.

2°. The proof in case \( \Gamma \neq \text{Ex}(\nu) \) is essentially the same as the proof of Proposition 3 in [13]. Since both \( \Gamma \) and \( \text{Ex}(\nu) \) are closed, \( C = \Gamma \setminus \text{Ex}(\nu) \) is relatively open in \( \Gamma \) and therefore it is not polar (see Proposition 1.1 and the definition of a normal pair). For the same reason, for every \( x \in C \) and every neighborhood \( U_\varepsilon \) of \( x \), \( \Gamma_\varepsilon(x) = U_\varepsilon \cap C \) is not polar.

Put

\[
\text{Cap}_{R,\nu}(B) = P_c\{R \cap B \neq \emptyset, e^{-Z_v}\}
\]

Likewise \( \text{Cap}_R \), it is a Choquet capacity by [1], Theorem III.32. Note that

\[
\text{Cap}_{R,\nu}(B) = e^{-u_{\nu}(c)} - e^{-w_{\nu}(c)}.
\]

(2.8)

By Lemma 2.5,

\[
P_c\{R \cap B \neq \emptyset, \mathcal{R} \cap \text{Ex}(\nu) = \emptyset\} > 0
\]

for every non-polar \( B \) disjoint from \( \text{Ex}(\nu) \). In addition, \( Z_v < \infty \) on \( \{\mathcal{R} \cap \text{Ex}(\nu) = \emptyset\} \) by Lemma 2.6. Therefore \( \text{Cap}_{R,\nu}(B) > 0 \) for every non-polar \( B \) disjoint from \( \text{Ex}(\nu) \). In particular, \( \text{Cap}_{R,\nu}(\Gamma_\varepsilon(x)) > 0 \) for every \( x \in C \).
Let \( x_i \) be everywhere dense in \( C \). Since Cap_{R,\nu}(x_i) = 0, one can choose \( \varepsilon_i > 0 \) to have
\[
\sum \text{Cap}_{R,\nu}(\Gamma_{\varepsilon_i}(x_i)) < \text{Cap}_{R,\nu}(C)
\]  
(2.9)

Put \( B = \cup \Gamma_{\varepsilon_i}(x_i) \). By (2.9),
\[
\text{Cap}_{R,\nu}(B) < \text{Cap}_{R,\nu}(C) \leq \text{Cap}_{R,\nu}(\Gamma)
\]
and, by (2.8) and Lemma 2.1,
\[
w_{B,\nu} < w_{\Gamma,\nu}.
\]  
(2.10)

Since \((\Gamma, \nu)\) is a normal pair, the trace of \( w_{\Gamma,\nu} \) coincides with \((\Gamma, \nu)\) by Proposition 1.1. Let now \((\bar{\Gamma}, \bar{\mu})\) be the trace of \( w_{B,\nu} \). By Lemma 2.4, \( \bar{\Gamma} \subset \Gamma \) and \( \mu = \nu \).

Suppose \( A = \Gamma \setminus \bar{\Gamma} \) is not empty. Since \( A \) is relatively open in \( \Gamma \), it contains at least one of \( x_i \).

The set \( A \cap \Gamma_{\varepsilon_i}(x_i) \) is also relatively open in \( \Gamma \), and therefore it is not polar by the definition of normal pair. But,
\[
A \cap \Gamma_{\varepsilon_i}(x_i) \subset A \cap B,
\]
and therefore \( B \setminus \bar{\Gamma} \) is not polar, in contradiction with Lemma 2.4. Therefore \( \bar{\Gamma} = \Gamma \) and the two solutions \( w_{B,\nu} \) and \( w_{\Gamma,\nu} \) do not coincide and have the trace \((\Gamma, \nu)\).

### 2.6 Proof of Theorem 1.4

1°. Recall some notation from [7] and [3]. A point \( y \in \partial E \) is a singular point of a solution \( u \) if
\[
\int_0^\zeta u^{n-1}(\xi_s) \, ds = \infty \quad \Pi^y \text{-a.s.}
\]

Here \((\xi_t, \Pi^y)\) is the \( L \)-diffusion in \( E \) conditioned to exit from \( E \) at the point \( y \), and \( \zeta \) is its life time. The set of all singular points of \( u \) is denoted by \( \text{SG}(u) \).

Let \( \Gamma = \text{SG}(u_\nu) \). By [3], Theorem 1.1
\[
P_\nu Z_\eta e^{-Z_\nu} = 0
\]

for every measure \( \eta \in \mathcal{N}_1 \) concentrated on \( \Gamma \). Since \( Z_\nu < \infty \) a.s. by Lemma 2.6, this is possible only if \( Z_\eta = 0 \) a.s. and therefore \( \eta = 0 \). By [6], Theorem 1.2, this is equivalent to the polarity of \( \Gamma \).

2°. Let \( B \) be a Borel subset of \( \partial E \). As in [7], denote by \( u_B \) the supremum of all moderate solutions \( u \) such that \( \mu(B^c) = 0 \). For two solutions \( u_1, u_2 \), we define \( u_1 \oplus u_2 \) as the maximal solution dominated by \( u_1 + u_2 \). See [7] for more detail.

Since \( \Gamma \) is polar, \( u_\Gamma = 0 \) and \( u_\Gamma \oplus u_\nu = u_\nu \).

3°. The fine trace of a solution \( u \) is defined as a pair \((\Gamma, \mu)\) where \( \Gamma = \text{SG}(u) \) and \( \mu \) is the maximal measure such that \( \mu(\Gamma) = 0 \), \( \mu \) does not charge polar sets and \( u_{\mu} \leq u \). According to [7], Theorem 1.3, the fine trace of any solution has the following properties:

(A) (See [7], 1.10.A.) The set \( \Gamma \) is finely closed (that is, closed in fine topology introduced in [7]).

(B) (See [7], 1.10.B.) The measure \( \mu \) is a \( \sigma \)-finite measure on \( \Gamma^c \) not charging polar sets and such that \( \text{SG}(u_{\mu}) \subset \Gamma \).
Moreover (see [7], Theorem 1.4), if $(\Gamma, \mu)$ is any pair satisfying (A) and (B), then $v = u_\Gamma \oplus u_\mu$ has the fine trace $(\Gamma', \mu)$ where $\Gamma' = \text{SG}(v)$ differs from $\Gamma$ by a polar set. The set $\Gamma = \text{SG}(u_\nu)$ is finitely closed by [7], Theorem 1.3. Since $\Gamma$ is polar, $\nu$ does not charge $\Gamma$. By assumption, $\text{Ess}(\nu)$ is empty and therefore $\nu$ is $\sigma$-finite by Theorem 1.1(iv). Hence the pair $(\Gamma, \nu)$ satisfies (A) and (B) and $u_\nu = u_\Gamma \oplus u_\nu$ has the fine trace $(\text{SG}(u_\nu), \nu) = (\Gamma, \nu)$. Since $\Gamma$ is polar, the statement follows.

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References


