MOMENT ESTIMATES FOR SOLUTIONS OF LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY ANALYTIC FRACTIONAL BROWNIAN MOTION

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Submitted March 2, 10, accepted in final form September 14, 2010

AMS 2000 Subject classification: 60G15, 60H05, 60H10
Keywords: stochastic differential equations, fractional Brownian motion, analytic fractional Brownian motion, rough paths, Hölder continuity, Chen series

Abstract
As a general rule, differential equations driven by a multi-dimensional irregular path $\Gamma$ are solved by constructing a rough path over $\Gamma$. The domain of definition – and also estimates – of the solutions depend on upper bounds for the rough path; these general, deterministic estimates are too crude to apply e.g. to the solutions of stochastic differential equations with linear coefficients driven by a Gaussian process with Hölder regularity $\alpha < \frac{1}{2}$.
We prove here (by showing convergence of Chen’s series) that linear stochastic differential equations driven by analytic fractional Brownian motion \cite{6,7} with arbitrary Hurst index $\alpha \in (0,1)$ may be solved on the closed upper half-plane, and that the solutions have finite variance.

1 Introduction

Assume $\Gamma_t = (\Gamma_t(1), \ldots, \Gamma_t(d))$ is a smooth $d$-dimensional path, and $V_1, \ldots, V_d : \mathbb{R}^r \to \mathbb{R}^r$ be smooth vector fields. Then (by the classical Cauchy-Lipschitz theorem for instance) the differential equation driven by $\Gamma$

$$ dy(t) = \sum_{i=1}^{d} V_i(y(t))d\Gamma_t(i) $$

admits a unique solution with initial condition $y(0) = y_0$. The usual way to prove this is by showing (by a functional fixed-point theorem) that iterated integrals

$$ y_n(t) \to y_{n+1}(t) := y_0 + \int_0^t \sum_{i} V_i(y_n(s))d\Gamma_t(i) $$

converge when $n \to \infty$. 

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Expanding this expression to all orders yields formally for an arbitrary analytic function $f$

\[ f(y_t) = f(y_s) + \sum_{n=1}^{\infty} \sum_{1 \leq i_1, \ldots, i_n \leq d} \left[ V_{i_1} \cdots V_{i_n} f \right](y_s) \Gamma^n_{t_0}(i_1, \ldots, i_n), \quad (1.3) \]

where

\[ \Gamma^n_{t_0}(i_1, \ldots, i_n) := \int_s^t d\Gamma_{i_1}(i_1) \int_s^{t_1} d\Gamma_{i_2}(i_2) \cdots \int_s^{t_{n-1}} d\Gamma_{i_n}(i_n), \quad (1.4) \]

provided, of course, the series converges. By specializing to the identity function $f = \text{Id} : \mathbb{R}^r \to \mathbb{R}^r$, $x \to x$, one gets a series expansion for the solution $(y_t)$.

Let

\[ \delta_{V}^{N,t,s}(y_s) = y_s + \sum_{i=1}^{N} \sum_{1 \leq i_1, \ldots, i_n \leq d} \left[ V_{i_1} \cdots V_{i_n} \text{Id} \right](y_s) \Gamma^n_{t_0}(i_1, \ldots, i_n), \quad (1.5) \]

be the $N$-th order truncation of $(1.3)$. It may be interpreted as one iteration of the numerical Euler scheme of order $N$, which is defined by

\[ y^{Euler:D}_{t_k} := \delta_{V}^{N,t_k,t_{k-1}} \circ \ldots \circ \delta_{V}^{N,t_1,t_0}(y_0), \quad (1.6) \]

for an arbitrary partition $D = \{ 0 = t_0 < \ldots < t_n = T \}$ of the interval $[0, T]$. When $\Gamma$ is only $\alpha$-Hölder with $\frac{1}{N+1} < \alpha \leq \frac{1}{N}$, the iterated integrals $\Gamma^n_{i_1, \ldots, i_n}$, $n=2, \ldots, N$ do not make sense a priori and must be substituted with a geometric rough path over $\Gamma$. A geometric rough path over $\Gamma$ is a family

\[ \left( \Gamma^1_{t_0}(i_1), \Gamma^2_{t_0}(i_1, i_2), \Gamma^3_{t_0}(i_1, i_2, i_3), \ldots, \Gamma^n_{t_0}(i_1, \ldots, i_n) \right), \quad (1.7) \]

of functions of two variables such that: $\Gamma^1_{t_0} = \Gamma^1_s - \Gamma^1_s$ and satisfying a natural Hölder regularity condition, namely, $\sup_{t,s \in \mathbb{R}} \left| \Gamma^n_{t_0}(i_1, \ldots, i_n) \right| < \infty$, $k=1, \ldots, N$, along with two algebraic compatibility properties (Chen/multiplicativity and shuffle/geometricity properties) for which we refer e.g. to [2]. To such data one may associate a theory of integration along $\Gamma$, so that $(1.1)$, rewritten in its integral form, makes sense, see e.g. [2] or [3] for local solutions of differential equations in this setting.

In this article, we prove convergence of the series $(1.3)$ when the vector fields $V_i$ are linear and $\Gamma$ is analytic fBm (affBm for short). This process -- first defined in [7] --, depending on an index $\alpha \in (0, 1)$, is a complex-valued process, a.s. $\kappa$-Hölder for every $\kappa < \alpha$, which has an analytic continuation to the upper half-plane $\Pi^+ := \{ z = x + iy \mid x \in \mathbb{R}, y > 0 \}$. Its real part $(2\Re \Gamma, t \in \mathbb{R})$ has the same law as fBm with Hurst index $\alpha$. Trajectories of $\Gamma$ on the closed upper half-plane $\bar{\Pi}^+ = \Pi^+ \cup \mathbb{R}$ have the same regularity as those of fBm, namely, they are $\kappa$-Hölder for every $\kappa < \alpha$. As shown in [6], the regularized rough path -- constructed by moving inside the upper half-plane through an imaginary translation $t \to t + i \varepsilon$ -- converges in the limit $\varepsilon \to 0$ to a geometric rough path over $\Gamma$ for any $\alpha \in (0, 1)$, which makes it possible to produce strong, local pathwise solutions of stochastic differential equations driven by $\Gamma$ with analytic coefficients.

We do not enquire about the convergence of the series $(1.3)$ in the general case (as mentioned before, it diverges e.g. when $V$ is quadratic), but only in the linear case. One obtains, see section 3:

Main Theorem.
Let $V_1, \ldots, V_d$ be linear vector fields on $\mathbb{C}$. Then the series (1.3), associated to afBm $\Gamma$ with Hurst index $\alpha \in (0, 1)$, converges in $L^2(\Omega)$ on the closed upper half-plane $\Pi^+ = \Pi^+ \cup \mathbb{R}$. Furthermore, there exists a constant $C$ such that the solution $(Y_t)_{t \in \Pi^+}$, defined as the limit of the series, satisfies
\[
E|y_t - y_s|^2 \leq C|t - s|^{2\alpha}, \quad s, t \in \Pi^+.
\] (1.8)

The Main Theorem depends essentially on an explicit estimate of the variance of iterated integrals of $\Gamma$ proved in Lemma 2.2 below, which states the following:

**Lemma 2.2.**
There exists a constant $C'$ such that, for every $s, t \in \Pi^+ = \Pi^+ \cup \mathbb{R}$,
\[
\text{Var}_{\mathbb{R}}(i_1, \ldots, i_n) \leq \frac{(C'|t - s|)^{2\alpha}}{n!}.
\] (1.9)

**Notation.** Constants (possibly depending on $\alpha$) are generally denoted by $C, C', C_1, c_\alpha$ and so on.

## 2 Definition of afBm and first estimates

We briefly recall to begin with the definition of the analytic fractional Brownian motion $\Gamma$, which is a complex-valued process defined on the closed upper half-plane $\Pi^+$ [6]. Its introduction was initially motivated by the possibility to construct quite easily iterated integrals of $\Gamma$ by a contour deformation. Alternatively, its Fourier transform is supported on $\mathbb{R}_+$, which makes the regularization procedure in [8, 9] void.

**Proposition 2.1.** There exists a unique analytic Gaussian process $(\Gamma^\gamma_z, z \in \Pi^+)$ with the following properties (see [6] or [7] for its definition):

1. $\Gamma^\gamma$ is a well-defined analytic process on $\Pi^+$, with Hermitian covariance kernel
\[
E\Gamma_z^\gamma \Gamma_w^\gamma = 0, \quad E\Gamma_z^\gamma \Gamma_w^\gamma = \frac{\alpha(1 - 2\alpha)}{2 \cos \pi \alpha} (-i(z - \bar{w}))^{2\alpha - 2}. \tag{2.1}
\]

2. Let $\gamma : (0, 1) \to \Pi^+$ be any continuous path with endpoints $\gamma(0) = 0$ and $\gamma(1) = z$, and set $\Gamma_z = \int_0^1 \Gamma^\gamma_u \quad du$. Then $\Gamma$ is an analytic process on $\Pi^+$. Furthermore, as $z$ runs along any path in $\Pi^+$ going to $t \in \mathbb{R}$, the random variables $\Gamma_z$ converge almost surely to a random variable called again $\Gamma_t$.

3. The family $\{\Gamma_t; t \in \mathbb{R}\}$ defines a Gaussian centered complex-valued process, whose covariance function is given by:
\[
E[\Gamma_t, \Gamma_s] = \begin{cases} 0, & t \neq s, \\ \frac{e^{-i\pi \alpha \text{sgn}(s)|s|^{2\alpha}} + e^{i\pi \alpha \text{sgn}(t)|t|^{2\alpha}} - e^{i\pi \alpha \text{sgn}(t-s)|s-t|^{2\alpha}}}{4 \cos(\pi \alpha)}, & t = s. \end{cases}
\]

The paths of this process are almost surely $\kappa$-Hölder for any $\kappa < \alpha$.

4. Both real and imaginary parts of $\{\Gamma_t; t \in \mathbb{R}\}$ are (non independent) fractional Brownian motions indexed by $\mathbb{R}$, with covariance given by
\[
E[\text{Re} \Gamma_t, \text{Im} \Gamma_s] = -\tan \frac{\pi \alpha}{8} \left[ -\text{sgn}(s)|s|^{2\alpha} + \text{sgn}(t)|t|^{2\alpha} - \text{sgn}(t-s)|t-s|^{2\alpha} \right]. \tag{2.2}
\]

**Definition 2.2.** Let $Y_t := \text{Re} \Gamma_t$, $t \in \mathbb{R}_+$. More generally, $Y_t = (Y_t(1), \ldots, Y_t(d))$ is a vector-valued process with $d$ independent, identically distributed components.
The above results imply that \( Y_t \) is real-analytic on \( \mathbb{R}^+ \).

**Lemma 2.3.** The infinitesimal covariance function of \( Y_t \) is:

\[
EY_t'Y_t' = \frac{\alpha(1 - 2\alpha)}{4 \cos \pi \alpha} (s + t)^{2\alpha - 2}. \tag{2.3}
\]

**Proof.** Let \( X_t := \text{Im} \, \Gamma_{it} \). Since \( E \Gamma_{it} \Gamma_{it} = 0 \), \( (Y_s, s \geq 0) \) and \( (X_s, s \geq 0) \) have same law, with covariance kernel \( EY_tX_t = \frac{1}{2} \text{Re} \, \Gamma_{it} \bar{\Gamma}_{it} \). Hence

\[
E[Y_t'Y_t'] = \frac{1}{2} \text{Re} \, \Gamma_{it} \bar{\Gamma}_{it} = \frac{\alpha(1 - 2\alpha)}{4 \cos \pi \alpha} (s + t)^{2\alpha - 2}. \tag{2.4}
\]

Note that \( EY_t'Y_t' > 0 \). From this simple remark follows (see proof of a similar statement in [5] concerning usual fractional Brownian motion with Hurst index \( \alpha > 1/2 \)):

**Lemma 2.4.** Let \( Y^n_{ts}(i_1, \ldots, i_n), n \geq 2 \) be the iterated integrals of \( Y \). Then there exists a constant \( C > 0 \) such that

\[
\text{Var} Y^n_{ts}(i_1, \ldots, i_n) \leq C \frac{(C|t-s|)^{2\alpha}}{n!}. \tag{2.5}
\]

**Proof.** Let \( \Pi \) be the set of all pairings \( \pi \) of the set \( \{1, \ldots, 2n\} \) such that \( ((k_1, k_2) \in \pi) \Rightarrow (i_{k_1'} = i_{k_2'}) \), where \( k_1' = k_1 \) if \( k_1 \leq n \), \( k_1 - n \) otherwise, and similarly for \( k_2' \). By Wick's formula,

\[
\text{Var} Y^n_{ts}(i_1, \ldots, i_n) = \sum_{\pi \in \Pi} \left( \int_s^t dx_1 \ldots \int_s^{x_{n-1}} dx_n \right) \left( \int_s^t dx_{n+1} \ldots \int_s^{x_{2n-1}} dx_{2n} \right) \prod_{(k_1, k_2) \in \pi} E[Y_{x_{k_1}}'Y_{x_{k_2}}']. \tag{2.6}
\]

Since the process \( Y' \) is positively correlated, and \( \Pi \) is largest when all indices \( i_1, \ldots, i_n \) are equal, one gets \( \text{Var} Y^n_{ts}(i_1, \ldots, i_n) \leq \text{Var} Y^n_{ts}(1, \ldots, 1) \). On the other hand, \( Y^n_{ts}(1, \ldots, 1) = \frac{1}{n!} (Y_t - Y_s)^n \), hence

\[
\text{Var} Y^n_{ts}(1, \ldots, 1) = \frac{[\text{Var}(Y_t - Y_s)]^n}{(n!)^2} \cdot \frac{(2n)!}{2^n \cdot n!} \leq \frac{[2\text{Var}(Y_t - Y_s)]^n}{n!}. \tag{2.7}
\]

Now (assuming for instance \( 0 < s < t \))

\[
\text{Var}(Y_t - Y_s) = c_\alpha \int_s^t \int_s^t (u + v)^{2\alpha - 2} du dv \leq c_\alpha s^{2\alpha - 2}(t - s)^2 \leq c_\alpha (t - s)^{2\alpha} \tag{2.8}
\]

if \( \frac{s}{2} \leq s \leq t \), and

\[
\text{Var}(Y_t - Y_s) = \frac{c_\alpha}{2\alpha(2\alpha - 1)} \left[ (2t)^{2\alpha} + (2s)^{2\alpha} - 2(t + s)^{2\alpha} \right] \leq C t^{2\alpha} \leq C'(t - s)^{2\alpha} \tag{2.9}
\]

if \( s < t/2 \). Hence the result. □
3 Estimates for iterated integrals of $\Gamma$

The main tool for the study of $\Gamma$ is the use of contour deformation. Iterated integrals of $\Gamma$ are particular cases of analytic iterated integrals, see [7] or [6]. In particular, the following holds:

Lemma 3.1. Let $\gamma : (0, 1) \rightarrow \Pi^+$ be the piecewise linear contour with affine parametrization defined by:

(i) $\gamma([0, 1/3]) = [s, s + i\text{Re}(t - s)];$

(ii) $\gamma([1/3, 2/3]) = [s + i\text{Re}(t - s)], t + i\text{Re}(t - s)];$

(iii) $\gamma([2/3, 1]) = [t + i\text{Re}(t - s)], t].$

If $z = \gamma(x) \in \gamma([0, 1]),$ we let $\gamma_z$ be the same path stopped at $z,$ i.e. $\gamma_z = \gamma([0, x]),$ with the same parametrization. Then (letting $c_a = \frac{a(1 - 2a)}{2\cos \pi a})$

$$\text{Var}^\Pi_t(i_1, \ldots, i_n) = c_a^\alpha \sum_{\sigma \in \Sigma} \int_{\gamma} dz_1 \int_{\gamma} dz_2 (-i(z_1 - \bar{\omega}_{\sigma(1)})^{2a-2} \cdot \int_{\gamma} dz_3\int_{\gamma} dz_4 (-i(z_2 - \bar{\omega}_{\sigma(2)})^{2a-2} \cdots$$

$$\int_{\gamma} dz_n \int_{\gamma} dz_n (-i(z_n - \bar{\omega}_{\sigma(n)})^{2a-2} \cdots (3.1)$$

where $\Sigma$ is the subset of permutations of $\{1, \ldots, n\}$ such that $(i_j = i_k) \Rightarrow (\sigma(j) = \sigma(k)).$

Proof. Note first that, similarly to eq. (2.6),

$$\text{Var}^\Pi_t(i_1, \ldots, i_n) = \sum_{\sigma \in \Sigma} \left( \int_{\gamma} dz_1 \cdots \int_{\gamma} dz_n \int_{\gamma} dz_1 \cdots \int_{\gamma} dz_n \right)$$

$$\int_{\gamma} dz_1 \cdots \int_{\gamma} dz_n \int_{\gamma} dz_1 \cdots \int_{\gamma} dz_n \left[ \Gamma^i_{\gamma} \Gamma^i_{\gamma} \right] \int_{\gamma} dz_1 \cdots \int_{\gamma} dz_n \int_{\gamma} dz_1 \cdots \int_{\gamma} dz_n (3.2)$$

(the difference with respect to eq. (2.6) comes from the fact that contractions only operate between $\Gamma$'s and $\bar{\omega}$'s, since $E[\Gamma^i_x, \Gamma^i_x] = E[\bar{\omega}^i_{\gamma}, \bar{\omega}^i_{\gamma}] = 0$ by Proposition 2.1. Now the result comes from a deformation of contour, see [7].\)

Lemma 3.2. There exists a constant $C'$ such that, for every $s, t \in \Pi^+ = \Pi^+ \cup \mathbb{R},$

$$\text{Var}^\Pi_t(i_1, \ldots, i_n) \leq \frac{(C' |t - s|)^{2na}}{n!}. \quad (3.3)$$

Proof. We assume (without loss of generality) that $\text{Im} s \leq \text{Im} t.$ If $|\text{Im}(t - s)| \geq c\text{Re}|t - s|$ for some positive constant $c$ (or equivalently $|\text{Re}(t - s)| \leq c'|t - s|$ for some $0 \leq c' < 1$) then it is preferable to integrate along the straight line $[s, t] = \{z \in \mathbb{C} | z = (1 - u)s + ut, 0 \leq u \leq 1\}$ instead of $\gamma,$ and use the parametrization $y = \text{Im} z.$ If $z_1, z_2 \in [s, t], y_1 = \text{Im} z_1, y_2 = \text{Im} z_2,$ then $|(-i(z_1 - z_2))^{2a-2}| \leq C(y_1 + y_2)^{2a-2},$ hence $\text{Var}^\Pi_t(i_1, \ldots, i_n) \leq C^n \text{Var}^\Pi_t(i_1, \ldots, i_n),$ which yields the result by Lemma 2.4. So we shall assume that $|\text{Re}(t - s)| > c|t - s|$ for some constant $c > 0.$
Let us use as new variable the parametrization coordinate $x$ along $\gamma$. Then formula (3.1) reads

$$\text{Var}^n_{\gamma^n}(i_1, \ldots, i_n) = c^n \sum_{a \in \mathcal{A}_2} \int_0^1 dx_1 \int_0^1 dy_1 K'(x_1, y_{\alpha(1)}) \cdot \int_0^{x_1} dx_2 \int_0^{y_1} dy_2 K'(x_2, y_{\alpha(2)}) \ldots \int_0^{x_{n-1}} dx_n \int_0^{x_n} dy_n K'(x_n, y_{\alpha(n)}),$$

where $K'(x, y) = (\text{Re} (t - s))^{\gamma(3(x + y) \text{Re} (t - s) + 2\text{Im} s)^{2a-2} \text{if } 0 < x, y < 1/3, (\text{Re} (t - s))^{\gamma((1 - x) + (1 - y))\text{Re} (t - s) + 2\text{Im} t)^{2a-2} \text{if } 2/3 < x, y < 1, and is bounded by a constant times $|t-s|^{2a}$ otherwise thanks to the condition $|Re (t-s)| > c|t-s|$. Note that $(x+y)^{2a} > 2^{2a-2}$ if $0 < x, y < 1$. Hence (if $0 < x, y < 1$) $|K'(x, y)| \leq (C_1|t-s|)^{2a} [(x+y)^{2a} + ((1-x) + (1-y))^{2a-2}]$, which is (up to a coefficient) the infinitesimal covariance of $|t-s|^a (Y_s + Y_{t-s}, 0 < x < 1)$ if $Y$ is independent of $Y$. A slight modification of the argument of Lemma 2.4 yields

$$\text{Var}^n_{\gamma^n}(i_1, \ldots, i_n) \leq (C_1|t-s|)^{2a} \left( \text{Var}(Y_s - Y_0) + \text{Var}(\tilde{Y}_i - \tilde{Y}_0) \right)^{n} \left( \frac{2n!}{n!} \right) \cdot \frac{(2n)!}{2^n \cdot n!} \leq C^n \cdot \frac{(2n|t-s|)^{2an}}{n!}.$$

(3.5)

\[\square\]

### 4 Proof of main theorem

We now prove the theorem stated in the introduction, which is really a simple corollary of Lemma 3.2

Let $C$ be the maximum of the matrix norms $||V||_\infty = \sup_{|x||x||_\infty 1} ||Vx||_\infty$ for the supremum norm $||x||_\infty = \sup(|x_1|, \ldots, |x_1|)$. Rewrite eq. (1.5) as

$$a_{i_1, \ldots, i_n}^n = y_s + \sum_{n=1}^N \sum_{1 \leq i_1, \ldots, i_n \leq d} a_{i_1, \ldots, i_n}^{\gamma^n}(i_1, \ldots, i_n).$$

(4.1)

Then $||a_{i_1, \ldots, i_n}||_\infty \leq C^n$ and $E|a_{i_1, \ldots, i_n}^n| \leq C^n |t-s|^{ma}$. Hence (by the Cauchy-Schwarz inequality)

$$E\left( a_{i_1, \ldots, i_n}^{\gamma^n}(y_s) - y_s \right)^2 \leq \sum_{m=1}^N \left( \frac{C^n |t-s|^{ma}}{\sqrt{m!n!}} \right)^2 \leq C^n |t-s|^{2a}$$

(4.2)

independently of $N$. The series obviously converges and yields eq. (1.8) for $p = 1$. It should be easy to prove along the same lines that the series defining $E|y_t - y_s|^{2p}$ converges for every $p \geq 1$, and that there exists a constant $C_p$ such that $E|y_t - y_s|^{2p} \leq C_p |t-s|^{2ap}$ for every $s, t \in \Pi^+$. The most obvious consequence – using Kolmogorov's lemma – would be that $y_s$ has Hölder regularity of any order less than $\alpha$. But this follows from standard rough path theory, so we skip the proof.
References


