GIRTH AND SUBDOMINANT EIGENVALUES FOR STOCHASTIC MATRICES

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Abstract. The set $S(g,n)$ of all stochastic matrices of order $n$ whose directed graph has girth $g$ is considered. For any $g$ and $n$, a lower bound is provided on the modulus of a subdominant eigenvalue of such a matrix in terms of $g$ and $n$, and for the cases $g = 1, 2, 3$ the minimum possible modulus of a subdominant eigenvalue for a matrix in $S(g,n)$ is computed. A class of examples for the case $g = 4$ is investigated, and it is shown that if $g > 2n/3$ and $n \geq 27$, then for every matrix in $S(g,n)$, the modulus of the subdominant eigenvalue is at least $(\frac{1}{5})^{1/(2[\frac{n}{3}])}$.

Key words. Stochastic matrix, Markov chain, Directed graph, Girth, Subdominant eigenvalue.

AMS subject classifications. 15A18, 15A42, 15A51.

1. Introduction and preliminaries. Suppose that $T$ is an irreducible stochastic matrix. It is well known that the spectral radius of $T$ is 1, and that in fact 1 is an eigenvalue of $T$ (with the all ones vector 1 as a corresponding eigenvector). Indeed, denoting the directed graph of $T$ by $D$ (see [2]), Perron-Frobenius theory (see [8]) gives more information on the spectrum of $T$, namely that the number of eigenvalues having modulus 1 coincides with the greatest common divisor of the cycle lengths in $D$. In particular, if that greatest common divisor is 1, it follows that the powers of $T$ converge. (This in turn leads to a convergence result for the iterates of a Markov chain with transition matrix $T$.) Denoting the eigenvalues of $T$ by $1 = \lambda_1(T) \geq |\lambda_2(T)| \geq \ldots \geq |\lambda_n(T)|$ (throughout we will use this convention in labeling the eigenvalues of a stochastic matrix), it is not difficult to see that the asymptotic rate of convergence of the powers of $T$ is governed by $|\lambda_2(T)|$. We refer to $\lambda_2(T)$ as a subdominant eigenvalue of $T$.

In light of these observations, it is natural to wonder whether stronger hypotheses on the directed graph $D$ will yield further information on the subdominant eigenvalue(s) of $T$. This sort of question was addressed in [6], where it was shown that if $T$ is a primitive stochastic matrix of order $n$ whose exponent (i.e. the smallest $k \in \mathbb{N}$ so that $T^k$ has all positive entries) is at least $\lfloor \frac{n^2 - 2n + 2}{2} \rfloor + 2$, then $T$ has at least $2[\frac{n - 4}{4}]$ eigenvalues with moduli exceeding $(\frac{1}{2}\sin[\pi/(n - 1)])^{2/(n-1)}$. Thus a hypothesis on the directed graph $D$ can lead to information about the eigenvalues of $T$.

In this paper, we consider the influence of the girth of $D$ - that is, the length of the shortest cycle in $D$ - on the modulus of the subdominant eigenvalue(s) of $T$. (It is straightforward to see that the girth of $D$ is the smallest $k \in \mathbb{N}$ such that $\text{trace}(T^k) > 0$.) Specifically, let $S(g,n)$ be the set of $n \times n$ stochastic matrices having

* Received by the editors 27 August 2004. Accepted for publication 2 December 2004. Handling Editor: Abraham Berman.
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digraphs with girth $g$. If $T \in S(g,n)$, how large can $|\lambda_2(T)|$ be? How small can $|\lambda_2(T)|$ be?

We note that the former question is readily dealt with. If $g \geq 2$, consider the directed graph $G$ on $n$ vertices that consists of a single $g$-cycle, say on vertices $1, \ldots, g$, along with a directed path $n \to n-1 \to \ldots \to g+1 \to 1$. Letting $A$ be the $(0,1)$ adjacency matrix of $G$, it is straightforward to determine that $A \in S(g,n)$, and that the eigenvalues of $A$ consist of the $g$-th roots of unity, along with the eigenvalue 0 of algebraic multiplicity $n-g$. In particular, $|\lambda_2(A)| = 1$, so we find that $\max\{|\lambda_2(T)||T \in S(g,n)\} = 1$. Similarly, for the case $g = 1$, we note that the identity matrix of order $n$, $I_n$, is an element of $S(1,n)$, and again we have $\max\{|\lambda_2(T)||T \in S(1,n)\} = 1$.

The bulk of this paper is devoted to a discussion of how small $|\lambda_2(T)|$ can be if $T \in S(g,n)$ (and hence, of how quickly the powers of $T$ can converge). To that end, we let $\lambda_2(g,n)$ be given by $\lambda_2(g,n) = \inf\{|\lambda_2(T)||T \in S(g,n)\}$.

**Remark 1.1.** We begin by discussing the case that $g = 1$. Let $J$ denote the $n \times n$ all ones matrix, and observe that for any $n \geq 2$, the $n \times n$ matrix $\frac{1}{n}J$ has the eigenvalues 1 and 0, the latter with algebraic and geometric multiplicity $n-1$. It follows immediately that that $\lambda_2(1,n) = 0$.

Indeed there are many stochastic matrices yielding this minimum value for $\lambda_2$, of all possible admissible Jordan forms. To see this fact, let $M$ be any nilpotent Jordan matrix of order $n-1$. Let $v_1, \ldots, v_{n-1}$ be an orthonormal basis of the orthogonal complement of 1 in $\mathbb{R}^n$, and let $V$ be the $n \times (n-1)$ matrix whose columns are $v_1, \ldots, v_{n-1}$. We find readily that for all sufficiently small $\epsilon > 0$, the matrix $T = \frac{1}{n}J + \epsilon V MV^T$ is stochastic; further, the Jordan form for $T$ is given by $[1] \oplus M$, so that the Jordan structure of $T$ corresponding to the eigenvalue 0 coincides with that of $M$. Evidently for such a matrix $T$, the powers of $T$ converge in a finite number of iterations; in fact that number of iterations coincides with the size of the largest Jordan block of $M$.

The following elementary result provides a lower bound on $\lambda_2(g,n)$ for $g \geq 2$.

**Theorem 1.1.** Suppose that $g \geq 2$ and that $T \in S(g,n)$. Then $|\lambda_2(T)| \geq 1/(n-1)^{\frac{1}{g-1}}$. Equality holds if and only if $g = 2$ and the eigenvalues of $T$ are 1 (with algebraic multiplicity 1) and $\frac{1}{n-1}$ (with algebraic multiplicity $n-1$). In particular,

$$\lambda_2(g,n) \geq 1/(n-1)^{\frac{1}{g-1}}.$$  

(1.1)

**Proof.** Let the eigenvalues of $T$ be $1, \lambda_2, \ldots, \lambda_n$. Since $\text{trace}(T^{g-1}) = 0$, we find that $\sum_{i=1}^{n} \lambda_i^{g-1} = -1$. Hence, $(n-1)|\lambda_2|^{g-1} \geq \sum_{i=2}^{n} |\lambda_i|^{g-1} \geq \sum_{i=2}^{n} \lambda_i^{g-1} = 1$. The inequality on $|\lambda_2|$ now follows readily.

Now suppose that $|\lambda_2| = 1/(n-1)^{\frac{1}{g-1}}$. Inspecting the proof above, we find that $|\lambda_i| = |\lambda_2|$, $i = 3, \ldots, n$, and that since equality holds in the triangle inequality, it must be the case that each of $\lambda_2, \ldots, \lambda_n$ has the same complex argument. Thus $\lambda_i = \lambda_1$ for each $i = 3, \ldots, n$. Since $\text{trace}(T) = 0$, we deduce that $\lambda_2 = -1/(n-1)$; but then $\text{trace}(T^2) = n/(n-1) > 0$, so that $g = 2$. The converse is straightforward.$\blacksquare$
REMARK 1.2. If $T \in S(2, n)$ and $|\lambda_2(T)| = 1/(n - 1)$, it is straightforward to see that the matrix $S = \frac{n-1}{2}T + \frac{1}{2}I_n$ has just two eigenvalues, 1 and 0, the latter with algebraic multiplicity $n - 1$. In particular, $S$ is a matrix in $S(1, n)$ such that $\lambda_2(S) = \lambda_2(1, n) = 0$.

REMARK 1.3. From Theorem 1.1, we see that if $\exists c > 0$ such that $g \geq cn$, then necessarily $\lambda_2(g, n) \geq 1/(n - 1 - \epsilon)$. An application of l'Hospital’s rule shows that $1/(n - 1 - \epsilon) \rightarrow 1$ as $n \rightarrow \infty$. Consequently, we find that for each $c > 0$, and any $\epsilon > 0$, there is a number $N$ such that if $n > N$ and $g \geq cn$, then each matrix $T \in S(g, n)$ has $|\lambda_2(T)| \geq 1 - \epsilon$.

We close this section with a discussion of $\lambda_2(g, n)$ as a function of $g$ and $n$.

PROPOSITION 1.2. Fix $g$ and $n$ with $2 \leq g \leq n - 1$. Then

a) $\lambda_2(g, n) \geq \lambda_2(g, n + 1)$, and

b) $\lambda_2(g + 1, n) \geq \lambda_2(g, n)$.

Proof. a) Suppose that $T \in S(g, n)$, and partition off the last row and column of $T$, say $T = \begin{bmatrix} T_1 & x \\ y^T & 0 \end{bmatrix}$. Now let $S$ be the stochastic matrix of order $n + 1$ given by

$S = \begin{bmatrix} T_1 & x \\ y^T & 0 \end{bmatrix}$. Note that the digraph of $S$ is formed from that of $T$ by adding the vertex $n + 1$, along with the arcs $i \rightarrow n + 1$ for each $i$ such that $i \rightarrow n$ in the digraph of $T$, and the arcs $n + 1 \rightarrow j$ for each $j$ such that $n \rightarrow j$ in the digraph of $T$. It now follows that the girth of the digraph of $S$ is also $g$, so that $S \in S(g, n + 1)$. Observe also that we can write $S$ as $S = ATB$, where the $(n + 1) \times n$ matrix $A$ is given by

$A = \begin{bmatrix} I_{n-1} & 0 \\ 0' & 1 \end{bmatrix}$, while the $n \times (n + 1)$ matrix $B$ is given by

$B = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0' & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$.

It is straightforward to see that $BA = I_n$: from this we find that since the matrix $ATB$ and the matrix $TBA$ have the same nonzero eigenvalues, so do $S$ and $T$. In particular, $\lambda_2(S) = \lambda_2(T)$, and we readily find that $\lambda_2(g, n) \geq \lambda_2(g, n + 1)$.

b) Let $\epsilon > 0$ be given, and suppose that $T \in S(g + 1, n)$ is such that $|\lambda_2(T)| < \lambda_2(g + 1, n) + \epsilon/2$. Without loss of generality, we suppose that the digraph of $T$ contains the cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots \rightarrow g + 1 \rightarrow 1$. For each $x \in (0, T_{g+1,g+1})$, let $S(x) = T + xe_1(e_1 - e_{g+1})^T$, where $e_i$ denotes the $i$-th standard unit basis vector. Note that for each $x \in (0, T_{g+1,g+1})$, $S(x) \in S(g, n)$. By the continuity of the spectrum, there is a $\delta > 0$ such that for any $0 < x < \min\{\delta, T_{g+1,g+1}\}$, $|\lambda_2(S(x)) - |\lambda_2(T)| < \epsilon/2$. Hence we find that for $0 < x < \min\{\delta, T_{g+1,g+1}\}$, $\lambda_2(g, n) \leq |\lambda_2(S(x))| < |\lambda_2(T)| + \epsilon/2 < \lambda_2(g + 1, n) + \epsilon$. In particular, we find that for each $\epsilon > 0$, $\lambda_2(g, n) \leq \lambda_2(g + 1, n) + \epsilon$, from which we conclude that $\lambda_2(g, n) \leq \lambda_2(g + 1, n)$.

2. Girths 2 and 3. In this section, we use some elementary techniques to find $\lambda_2(2, n)$ and $\lambda_2(3, n)$. We begin with a discussion of the former.

THEOREM 2.1. For any $n \geq 2$, $\lambda_2(2, n) = 1/(n - 1)$.

Proof. From Theorem 1.1, we have $\lambda_2(2, n) \geq 1/(n - 1)$; the result now follows upon observing that the matrix $\frac{1}{n-1}(J - I) \in S(2, n)$, and has eigenvalues 1 and $-1/(n - 1)$, the latter with multiplicity $n - 1$. □
Our next result shows that there is just one diagonable matrix that yields the minimum value $\lambda_2(2, n)$.

**Theorem 2.2.** Suppose that $T \in S(2, n)$. Then $T$ is diagonable with $|\lambda_2(T)| = 1/(n - 1)$ if and only if $T = \frac{1}{n-1}(J - I)$.

**Proof.** Suppose that $T$ is diagonable, with $|\lambda_2(T)| = 1/(n - 1)$; from Theorem 1.1 we find that the eigenvalue $\lambda_2 = -1/(n - 1)$ has algebraic multiplicity $n - 1$. Since $T$ is diagonable, the dimension of the $\lambda_2$-eigenspace is $n - 1$. Let $x^T$ be the left Perron vector for $T$, normalized so that $x^T1 = 1$. It follows that there are right $\lambda_2$-eigenvectors $v_2, \ldots, v_n$ and left $\lambda_2$-eigenvectors $w_2, \ldots, w_n$, so that $T = 1x^T + \frac{1}{n-1}\sum_{i=2}^n v_iw_i^T$ and $I = 1x^T + \sum_{i=2}^n v_iw_i^T$. Substituting, we see that $T = \frac{1}{n-1}(n1x^T - I)$, and since $T$ has trace zero, necessarily, $x^T = \frac{1}{n}1^T$, yielding the desired expression for $T$. The converse is straightforward. \[\square\]

Our next example shows that other Jordan forms are possible for matrices yielding the minimum value $\lambda_2(2, n)$.

**Example 2.1.** Consider the polynomial

$$(\lambda + \frac{1}{n-1})^{n-1} = \sum_{j=0}^{n-1} \lambda^j \left( \frac{1}{n-1} \right)^{n-1-j} \binom{n-1}{j}$$

$$= \lambda^{n-1} + \lambda^{n-2} + \sum_{j=0}^{n-3} \lambda^j \left( \frac{1}{n-1} \right)^{n-1-j} \binom{n-1}{j}.$$ 

From the fact that $n - j > \frac{j}{n-1}$ for $j = 1, \ldots, n - 2$, it follows readily that

$$(\frac{1}{n-1})^{n-1-j} \binom{n-1}{j} > (\frac{1}{n-1})^{n-1} \binom{n-1}{j}$$

for each such $j$.

We thus find that $(\lambda - 1)(\lambda + \frac{1}{n-1})^{n-1}$ can be written as $\lambda^n - \sum_{j=2}^n a_j\lambda^{n-j}$, where $a_j > 0$ for $j = 2, \ldots, n$, and $\sum_{j=2}^n a_j = 1$. Consequently, the companion matrix

$$C = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & 0
\end{bmatrix}$$

is in $S(2, n)$, and $\lambda_2(C) = -1/(n - 1)$. Note that since any eigenvalue of a companion matrix is geometrically simple, the eigenvalue $-1/(n - 1)$ of $C$ has a single Jordan block of size $n - 1$.

Next, we compute $\lambda_2(3, n)$ for odd $n$.

**Theorem 2.3.** Suppose that $n \geq 3$ is odd. If $T \in S(3, n)$, then $|\lambda_2(T)| \geq \frac{1}{n-2}$, with equality holding if and only if the eigenvalues of $T$ are 1 (with algebraic multiplicity one) and $\frac{1-\sqrt{5}}{n-2}$ (with algebraic multiplicity $(n - 1)/2$ each). Further, $\lambda_2(3, n) = \frac{1}{n-2}$.

**Proof.** Suppose that $T \in S(3, n)$, and denote the eigenvalues of $T$ by 1, and $x_j + iy_j$, $j = 2, \ldots, n$ (where of course each complex eigenvalue appears with a corresponding complex conjugate). Since $\text{trace}(T^2) = 0$, we have $\sum_{j=2}^n x_j = -1$, while from the fact that $\text{trace}(T^2) = 0$, we have $1 + \sum_{j=2}^n (x_j^2 - y_j^2) = 0$. Consequently,
that \( (\lambda - \frac{1}{n-1})^{n-1} \) is nonnegative, it suffices to show that the coefficients of the 
\[ \lambda \]

is a companion matrix

order

necessarily has girth 3.

digraph of a tournament matrix, and a standard result in the area asserts that the

finiteness \( (\lambda - \frac{1}{n-1})^{n-1} \) is a companion matrix

\[ q(\lambda) = \left( (\lambda + \frac{1}{n-1})^2 + \frac{n-2}{(n-1)^2} \right) \]

Note that \( q(\lambda) = \left( (\lambda + \frac{1}{n-1})^2 + \frac{n-2}{(n-1)^2} \right) \)

Applying the binomial expansion, and collecting powers of \( \lambda \), we find that

\[ (n-1)^2/j \]

Write \( q(\lambda) = \sum_{l=0}^{n-1} \lambda^l \alpha_l \). We claim that \( \alpha_l \geq \alpha_{l-1} \) for each \( l = 1, \ldots, n-1 \), which will yield the desired result. Note that for each such \( l \), the inequality \( \alpha_l \geq \alpha_{l-1} \) is equivalent to \( (n-1) \sum_{j=[l/2]}^{n-1} \left( \binom{n-1}{j} - \binom{n-1}{l/2} \right) \geq 0 \). Observe that

Finally, suppose that \( l \) is odd with \( 1 \leq l \leq n-1 \) and \( l = 2r + 1 \). Then \([l/2] = r + 1, \lfloor(l-1)/2\rfloor = r\), and since \( 2r + 1 \leq n-1 \), we find that \( r \leq \frac{n-3}{2} \). In order to show that \( \alpha_l \geq \alpha_{l-1} \), it suffices to show, in conjunction with the inequalities proven above, that

\[ \frac{n-1}{2} \approx \frac{n-3}{2} \approx 0 \]

Inequality can be seen to be equivalent to \( 2\frac{n^2-1}{n} - 2\frac{n-1}{n-2} \geq 0 \), and since we have \( 2\frac{n^2-1}{n} - 2\frac{n-1}{n-2} \geq 2\frac{n^2-1}{n} - \frac{2n^2-2}{n-1} \geq 0 \), the desired inequality is thus established. Hence for odd \( l \), we have \( \alpha_l \geq \alpha_{l-1} \), and it now follows that there is a companion matrix \( C \in S(3, n) \) such that \( |\lambda_2(C)| = \frac{\sqrt{n+1}}{n-1} \).

**Example 2.2.** Another class of matrices in \( S(3, n) \) yielding the minimum value for \( |\lambda_2| \) arises in the following combinatorial context. A square \((0, 1)\) matrix \( A \) of order \( n \) is called a tournament matrix if it satisfies the equation \( A + A^T = J - I \). From that equation, one readily deduces that there are no cycles of length 2 in the digraph of a tournament matrix, and a standard result in the area asserts that the digraph associated with any tournament matrix either contains a cycle of length 3, or it has no cycles at all. Thus the digraph of any nonnilpotent tournament matrix necessarily has girth 3.

If, in addition, a tournament matrix \( A \) satisfies the identity \( A^T A = \frac{n+1}{4} I + \frac{n-1}{4} J = A A^T \), then \( A \) is known as a doubly regular (or Hadamard) tournament ma-
trix; note that necessarily \( n \equiv 3 \mod 4 \) in that case. It turns out that doubly regular tournament matrices are co-existent with skew-Hadamard matrices, and so of course the question of whether there is a doubly regular tournament matrix in every admissible order is open, and apparently quite difficult.

In [3] it is shown that if \( A \) is a doubly regular tournament matrix, then its eigenvalues consist of \( \frac{n-1}{2} \) (of algebraic multiplicity one, and having 1 as a corresponding right eigenvector) and \( \frac{1}{2} \pm i \frac{\sqrt{n}}{2} \), each of algebraic multiplicity \( (n-1)/2 \). Consequently, we find that if \( A \) is an \( n \times n \) doubly regular tournament matrix, then \( T = \frac{2}{n-1} A \) is in \( S(3, n) \) and has eigenvalues 1 and \( \frac{2}{n-1} \pm \frac{2}{n} \sqrt{n^2+n+2} \), the latter with algebraic multiplicity \( (n-1)/2 \) each. From Theorem 2.3, we find that \( |\lambda_2(T)| = \rho_2(3, n) \), which is an \( n \times n \) doubly regular tournament matrix co-existent with skew-Hadamard matrices, and so of course \( n \equiv 3 \mod 4 \) in that case. It turns out that doubly regular tournament matrices are co-existent with skew-Hadamard matrices, and so of course the question of whether there is a doubly regular tournament matrix in every admissible order is open, and apparently quite difficult.

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We adapt the technique of the proof of Theorem 2.3 in order to compute \( \lambda_2(3, n) \) for even \( n \).

**Theorem 2.4.** Suppose that \( n \geq 4 \) is even. If \( T \in S(3, n) \), then \( |\lambda_2(T)| \geq \sqrt{\frac{n^2+n+2}{n-2}} \), with equality holding if and only if the eigenvalues of \( T \) are 1 (with algebraic multiplicity one), \( -2/n \) (also with algebraic multiplicity one) and \( \frac{1}{n} \pm \frac{1}{n} \sqrt{\frac{n^2+n+2}{n-2}} \) (with algebraic multiplicity \( (n-2)/2 \) each). Further, \( \lambda_2(3, n) = \sqrt{\frac{n^2+n+2}{n-2}} \).

**Proof.** Suppose that \( T \in S(3, n) \). Since \( T \) is stochastic, it has 1 as an eigenvalue, and since \( n \) is even, there is at least one more real eigenvalue for \( T \), say \( z \). Let \( x_j + iy_j, j = 2, \ldots, n-1 \), denote the remaining eigenvalues of \( T \). From the fact that \( \text{trace}(T) = 0 \), we have \( 1 + z + \sum_{j=2}^{n-1} x_j = 0 \), while \( \text{trace}(T^2) = 0 \) yields \( 1 + z^2 + \sum_{j=2}^{n-1} (x_j^2 - y_j^2) = 0 \). Thus we have \( \sum_{j=2}^{n-1} (x_j^2 + y_j^2) = 1 + z^2 + 2 \sum_{j=2}^{n-1} x_j^2 \). Consequently, we find that \( (n-2)|\lambda_2^2| \geq \sum_{j=2}^{n-1} (x_j^2 + y_j^2) = 1 + z^2 + 2 \sum_{j=2}^{n-1} x_j^2 \geq 1 + z^2 + 2(1+z^2)/(n-2) \), the second inequality following from the Cauchy-Schwarz inequality. The expression \( 1 + z^2 + 2(1+z^2)/(n-2) \) is readily seen to be uniquely minimized when \( z = -2/n \), with a minimum value of \( \frac{n^2+n+2}{n} \). Hence we find that \( (n-2)|\lambda_2^2| \geq \frac{n^2+n+2}{n} \), and the lower bound on \( |\lambda_2| \) follows.

Inspecting the argument above, we see that if \( |\lambda_2(T)| = \sqrt{\frac{n^2+n+2}{n-2}} \), then necessarily \( z \) must be \(-2/n \), each \( x_j \) must be \(-1/n \), while each \( y_j^2 \) is equal to \( \frac{1}{n^2} \sqrt{n^2+n+2} \). The characterization of equality now follows.

We claim that for each even \( n \), there is a companion matrix in \( S(3, n) \) having \( \frac{n^2+n+2}{n-2} \) as a subdominant eigenvalue. To see the claim, first consider the polynomial \( q(\lambda) = \left( \lambda - \frac{1}{n} \sqrt{\frac{n^2+n+2}{n-2}} \right)^{(n-2)/2} \left( \lambda - \left( \frac{1}{n} + \frac{1}{n} \sqrt{\frac{n^2+n+2}{n-2}} \right)^{(n-2)/2} \right) \). Since \( \lambda \) is the only real root of this polynomial, we may write it as \( q(\lambda) = \sum_{l=0}^{n-2} \lambda^l a_l \), so that \( (\lambda + 2/n)q(\lambda) = \lambda^{n-1} + \sum_{l=1}^{n-2} \lambda^l (a_{l-1} + 2a_l/\lambda) + 2a_0/\lambda \). As in the proof of Theorem 2.3, it suffices to show that in this last expression, the coefficients of \( \lambda^l \) are nondecreasing in \( l \). However, the coefficients of \( \lambda^l \) in this last expression are nondecreasing in \( l \). Also as in the proof of that theorem, we find that for each \( l = 0, \ldots, n-2, a_l = \sum_{j=[l/2]}^{n-1} \frac{1}{n} \lambda^{(n-2)/2 - j} \left( \frac{1}{n} \sqrt{\frac{n^2+n+2}{n-2}} \right)^{(n-2)/2 - j} \left( \frac{1}{n} \sqrt{\frac{n^2+n+2}{n-2}} \right)^{(n-2)/2} \). straightforward computations now reveal that the coefficients of \( \lambda^{n-1}, \lambda^{n-2}, \lambda^{n-3} \) and \( \lambda^{n-4} \) in the polynomial \( (\lambda + 2/n)q(\lambda) \) are nondecreasing in \( l \).
2/n)q(λ) are 1, 1, and \( \frac{2n^2-3n-2}{3n^2} \), respectively. We claim that for each \( l = 1, \ldots, n-4 \), \( a_l \geq a_{l-1} \), which is sufficient to give the desired result.

The claim is equivalent to proving that for each \( l = 1, \ldots, n-4 \),
\[
n \sum_{j=\lfloor l/2 \rfloor}^{(n-2)/2} \binom{n-2}{j} \binom{(n-2)/2}{j} \geq \sum_{j=(l-1)/2}^{(n-2)/2} \binom{n-2}{j} \binom{(n-2)/2}{j}. \]
Observe that \( n \binom{2j}{j} - \binom{2j}{j-1} = \frac{2n!}{j!(j+1)!} \left( \frac{n}{j+1} - \frac{1}{j} \right) \geq 0 \), so in particular, if \( l \) is even (so that \(|l/2| = (l-1)/2\)) it follows readily that \( a_l \geq a_{l-1} \). Now suppose that \( l \geq 1 \) is odd, say \( l = 2r+1 \), so that \(|l/2| = r+1 \) and \(|l-1)/2| = r \). Note also that since \( l \leq n-4 \), in fact \( l \leq n-5 \), so that \( r \leq (n-6)/2 \). In conjunction with the argument above, it suffices to show that
\[
n \sum_{j=\lfloor 2l/3 \rfloor}^{n/2} \binom{n}{2j} \binom{n/2}{j} \geq \sum_{j=\lfloor l/2 \rfloor}^{(n-2)/2} \binom{n-2}{j} \binom{(n-2)/2}{j} - \binom{n-2}{\lfloor l/2 \rfloor} \binom{n/2}{\lfloor l/2 \rfloor} \geq 0. \]
This last inequality can be seen to be equivalent to
\[
\frac{2\binom{n}{2l-2}}{n^2+2l-2} - \left( 2l + \frac{1}{2} \right) \frac{n^2+2l-2}{n^2+2l-3} \geq 0.
\]
\( \frac{2\binom{n}{2l-2}}{n^2+2l-2} - \left( 2l + \frac{1}{2} \right) \frac{n^2+2l-2}{n^2+2l-3} \geq \frac{2\binom{n}{2l-2}}{n^2+2l-2} - \frac{n^2+2l-2}{2(n^2+2l-3)} \geq 0, \)
the last since \( n \geq 4 \). Hence we have \( a_l \geq a_{l-1} \) for each \( l = 1,\ldots, n-4 \), as desired. \( \Box \)

The following result shows that the lower bound of (1.1) on \( \lambda_2(g,n) \) is of the correct order of magnitude for \( g = 3 \). Its proof is immediate from Theorems 2.3 and 2.4.

**Corollary 2.5.** \( \lim_{n \to \infty} \lambda_2(3, n) \sqrt{n} = 1 \).

**3. A class of examples for girth 4.** Our object in this section is to identify, for infinitely many \( n \), a matrix \( T \in S(4,n) \) such that \( \lambda_2(T) \) is of the same order of magnitude as \( 1/\sqrt{n-1} \), the lower bound on \( \lambda_2(4,n) \) arising from (1.1). Our approach is to identify a certain sequence of candidate spectra, and then show that each candidate spectrum is attained by an appropriate stochastic matrix.

Fix an integer \( p \geq 3 \), and let \( r = \frac{1}{3p} \). Set \( q = 9p^3 + 2p, l = 18p^3 + 9p^2 + p \) and \( m = 9p^2 + 3p \). Letting \( n = q + l + m + 1 \), it follows that \((n-1)r^4 - 2r^2 - 2r - 1 = 0 \). We would like to show that there is a matrix \( T \in S(4,n) \) whose eigenvalues are: 1 (with multiplicity 1), \(-r \) (with multiplicity \( q \)), \( re^{\pm \pi i/3} \) (each with multiplicity \( l/2 \)) and \( re^{\pm 2\pi i/3} \) (each with multiplicity \( m/2 \)).

For each \( j \in \mathbb{N} \), let
\[
s_j = 1 + q(-r)^{j+1} + (l/2)(re^{\pi i/3})^j + (l/2)(re^{-\pi i/3})^j + (m/2)(re^{2\pi i/3})^j + (m/2)(re^{-2\pi i/3})^j.
\]
(Observable that if we could find the desired matrix \( T \), then \( s_j \) would just be the trace of \( T^j \). We find readily that \( s_1 = s_2 = s_3 = 0 \), while \( s_4 = 1 - r^2, s_5 = 1 - r^4 \), and \( s_6 = 1 + r^3 + 2r^2 + 2r \). Finally, note that for any \( j \in \mathbb{N} \), \( s_{j+6} = 1 - r^6(s_j - 1) \).

Write the polynomial
\[
(\lambda - 1)(\lambda + r)^q(\lambda - re^{\pi i/3})^l(\lambda - re^{-\pi i/3})^l(\lambda - re^{2\pi i/3})^m(\lambda - re^{-2\pi i/3})^m
\]
as \( \lambda^n + \sum_{j=0}^{n-1} a_j \lambda^j \). Let \( C_n = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 1 \\
-a_0 & -a_1 & \cdots & -a_{n-1}
\end{bmatrix} \) be the asso-

to 1 + 

The offdiagonal entries of $M_k^{-1}$ are nonpositive, so that $M_k^{-1}$ is an $M$-matrix,
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b) $M^{-1}_k 1 \geq \frac{1}{k+1} 1$, and

c) $M^{-1}_k \begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3} \end{bmatrix}$ is a positive vector.

Proof: We proceed by extended induction on $k$ using a single induction proof for all three statements. Note that each of a), b) and c) is easily established for $k = 1, \ldots, 6$. Suppose now that a), b) and c) hold for natural numbers up to and including $k - 1 \geq 6$.

First, we consider statement a). We have $M^{-1}_k = \begin{bmatrix} 1/k \\ -y \\ M^{k-1}_k \end{bmatrix}$, where $y$ can be written as $y = \frac{1}{k}$

$M^{-1}_{k-4} \begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k-1} \end{bmatrix}$.

From part c) of the induction hypothesis, it follows that $y$ is a nonnegative vector, while from part a) of the induction hypothesis, the offdiagonal entries of $M^{-1}_{k-1}$ are also nonpositive. Hence all offdiagonal entries of $M^{-1}_k$ are nonpositive, which completes the proof of the induction step for statement a).

Next, we consider statement b). From Lemma 3.1, it follows that

$$M^{-1}_k 1 = \frac{1}{k-3-r^2} (1 - (3+r^2)M^{-1}_k e_1 - (2+r^2)M^{-1}_k e_2 - (1+r^2)M^{-1}_k e_3 - r^2 M^{-1}_k e_4 + r^3 M^{-1}_k \begin{bmatrix} 0 \\ e_k \end{bmatrix})$$

for some vector $v$ with $||v||_\infty = 1 + r + 2r^2$. The first four entries of $M^{-1}_k 1$ are $1/k, 1/(k-1), 1/(k-2)$ and $1/(k-3)$, respectively, so it remains only to show that $M^{-1}_k 1 \geq \frac{1}{k+1} 1$ in positions after the fourth.

Let $\text{trunc}_4(M^{-1}_k 1)$ denote the vector formed from $M^{-1}_k 1$ by deleting its first four entries. Noting that the entries of $M^{-1}_k e_1, M^{-1}_k e_2, M^{-1}_k e_3, M^{-1}_k e_4$ are nonpositive after the fourth position, it follows that $\text{trunc}_4(M^{-1}_k 1) \geq \frac{1}{k+3-r^2} 1 + r^3 \begin{bmatrix} 0 \\ M^{-1}_k e_k \end{bmatrix}$.

From part b) of the induction hypothesis, $M^{-1}_{k-6} 1$ is a positive vector, and from part a) of the induction hypothesis, $M^{-1}_{k-6}$ is an M-matrix. Note that $M^{-1}_{k-6}$ has diagonal entries $1/(k-6), 1/(k-7), \ldots, 1/2, 1$. Letting $u_i$ be the $i$-th row sum of $M^{-1}_{k-6}$, it follows that $||e_i^T M^{-1}_{k-6} 1||_\infty = 1/(k+5+i) + (1/(k-5+i) - u_i) \leq 2/(k-5+i) \leq 2$. Letting $||\bullet||_\infty$ denote the absolute row sum norm (induced by the infinity norm for vectors), we conclude that $||M^{-1}_{k-6} 1||_\infty \leq 2$. Hence $M^{-1}_{k-6} v \geq -2 ||v||_\infty 1 = -2(1 + r + 2r^2) 1$. As
a result, we have \[ \frac{1}{k-3-r^2} \mathbf{1} + \frac{r^3}{k-3-r^2} \left[ \frac{0_2}{M_{k-6}^\top} \right] \geq \frac{1}{k-3-r^2} \mathbf{1} - 2(1 + r + 2r^2) \frac{r^3}{k-3-r^2} \mathbf{1} = \frac{1 - 2r^3(1 + r + 2r^2)}{k-3-r^2} \mathbf{1}. \]

Since \((k-1)r^3 \leq 2r^2 + 2r + 1\), we have

\[ \frac{1 - 2r^3(1 + r + 2r^2)}{k-3-r^2} \geq \frac{1 - 2(1 + r + 2r^2)(1 + 2r + 2r^2)/(k-1)}{k-3-r^2} \geq \frac{k - 3.8325}{(k-1)(k-3)}, \]

the last inequality following from the fact that \(r \leq 1/9\). Since \(k \geq 7\), we find readily that \(\frac{k - 3.8325}{(k-1)(k-3)} \geq \frac{1}{k-1}\). Putting the inequalities together, we have \(M_k^{-1} \geq \frac{1}{k-1} \mathbf{1}\), which completes the proof of the induction step for statement b).

Finally, we consider statement c). We have \[
\begin{bmatrix}
-\frac{r^2}{e} \\
-\frac{r^4}{e} \\
0
\end{bmatrix} + \begin{bmatrix}
0_6 \\
-\frac{s_{10} - 1}{e} \\
\vdots \\
-\frac{s_{k+3} - 1}{e}
\end{bmatrix}.
\]
Recall that for \(4 \leq j \leq 9\) and \(i \in \mathbb{N}\),

\[ s_{j+6i} - 1 = r^6(s_j - 1), \]

so that \(\frac{s_{j+6i} - 1}{r^6} \leq \frac{s_j - 1}{r^6} \leq 1\). Hence

\[ \begin{bmatrix}
\frac{s_4}{e} \\
\frac{s_5}{e} \\
\vdots \\
\frac{s_{k+3}}{e}
\end{bmatrix} = \mathbf{1} - r^2 e_1 -
\]

\[ r^4 e_2 + r^3(1 + 2r + 2r^2)e_3 - r^6(e_4 + e_5 + e_6) + r^8 \left[ \frac{0_6}{e} \right], \]

where \(||v||_{\infty} \leq 1\). Thus we have

\[ M_k^{-1} \begin{bmatrix}
s_4 \\
s_5 \\
\vdots \\
s_{k+3}
\end{bmatrix} = M_k^{-1} \mathbf{1} - M_k^{-1}(r^2 e_1 + r^4 e_2 + r^6(e_4 + e_5 + e_6)) +
\]

\[ r^3(1 + 2r + 2r^2)M_k^{-1} e_3 + r^8 \left[ \frac{0_6}{M_k^{-1} e_3} \right]. \]

Certainly the first six entries of \(M_k^{-1} \begin{bmatrix}
s_4 \\
s_5 \\
\vdots \\
s_{k+3}
\end{bmatrix} \) are positive, so it remains only to show that the remaining entries are positive. Note also that the entries of \(M_k^{-1}(r^2 e_1 +
\]
As above, since $M_{k-6}^{-1}$ is an M-matrix, we find that $||M_{k-6}^{-1}||_\infty \leq 2$. Applying b), and using the bound on the norm of $M_{k-6}^{-1}$, we have

$$trunc_6(M_{k-1}^{-1}) - \frac{r^3(1 + 2r + 2r^2)}{k - 2} M_{k-6}^{-1} \begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k-3} \end{bmatrix} + r^8 M_{k-6}^{-1}v \geq 0$$

Thus, it is sufficient to show that

$$\frac{1}{k+1} - \frac{r^3(1 + 2r + 2r^2)}{k-2} - 2r^8 > 0.$$
The preceding results lead to the following.

**Theorem 3.3.** $M_n^{-1}A_n$ is an irreducible nonnegative matrix.

**Proof.** We claim that for each $4 \leq k \leq n$, $M_k^{-1}A_k$ is irreducible and nonnegative. The statement clearly holds if $k = 4$, and we proceed by induction. Suppose that the claim holds for some $4 \leq k \leq n - 1$. Note that $M_{k+1} = \begin{bmatrix} k + 1 & 0^T \\ s & M_k \end{bmatrix}$, where $s = \begin{bmatrix} s_1 \\ \vdots \\ s_k \end{bmatrix}$. We also have $A_{k+1} = \begin{bmatrix} 0 & kc^T \\ \sigma & A_k \end{bmatrix}$, where $\sigma = \begin{bmatrix} s_2 \\ \vdots \\ s_{k+1} \end{bmatrix}$. It then follows that $M_{k+1}^{-1}A_{k+1} = \begin{bmatrix} 0 & \frac{k}{\sigma} c^T \\ \frac{k}{\sigma} c \sigma^{-1} M_k^{-1} & M_k^{-1}A_k - \frac{k}{\sigma} \frac{c^T c}{M_k^{-1}} \end{bmatrix}$.

From the induction hypothesis, $M_k^{-1}A_ke_j \geq 0$ for each $1 \leq j \leq k$. Note also that $M_k^{-1}A_ke_1 = M_k^{-1}s \geq 0$, so that the first column of $M_k^{-1}A_k - \frac{k}{\sigma} \frac{c^T c}{M_k^{-1}}$ is just $\frac{1}{\sigma} M_k^{-1}s$, which is nonnegative, and has the same zero-nonzero pattern as the first column of $M_k^{-1}A_k$. Thus the $(2,2)$ block of $M_{k+1}^{-1}A_{k+1}$ is nonnegative and irreducible by the induction hypothesis, while the $(1,2)$ block is a nonnegative nonzero vector. Further, from Proposition 3.2 it follows that $M_k^{-1}\sigma$ is also nonnegative and nonzero. Hence $M_{k+1}^{-1}A_{k+1}$ is both nonnegative and irreducible, completing the induction step.

Here is the main result of this section; it follows from Theorem 3.3.

**Theorem 4.** For infinitely many $n$, $\lambda_2(4,n) \leq r$, where $r$ is the positive root of the equation $(n-1)r^3 - 2r^2 - 2r - 1 = 0$.

**Remark 3.1.** Let $f(x) = (n-1)x^3 - 2x^2 - 2x - 1$. A straightforward computation shows that for all sufficiently large $n$, $f((n-1)^{-\frac{1}{3}} + (n-1)^{-\frac{2}{3}}) > 0$. It now follows that for all sufficiently large $n$, the positive root $r$ for the function $f$ satisfies $r < (n-1)^{-\frac{1}{3}} + (n-1)^{-\frac{2}{3}}$.

The following is immediate from Theorem 1.1, Theorem 3.4 and Remark 3.1.

**Corollary 3.5.** $\liminf_{n \to \infty} \lambda_2(4,n) \sqrt{n} - 1 = 1$.

### 4. Bounds for large girth.

At least part of the motivation for the study of $\lambda_2(g,n)$ is to develop some insight when $g$ is large relative to $n$. As noted in Remark 1.3, if both $n$ and $g$ are large, then we expect $\lambda_2(g,n)$ to be close to $1$, so that any primitive matrix in $S(g,n)$ will give rise to a sequence of powers that converges only very slowly. The purpose of this section is to quantify these notions more precisely. To that end, we focus on the case that $g > 2n/3$.

The following result is useful. Its proof appears in [4] and (essentially) in [6] as well.

**Lemma 4.1.** Suppose that $g > n/2$ and that $T \in S(g,n)$. Then the characteristic polynomial for $T$ has the form $\lambda^n - \sum_{j=0}^n a_j \lambda^{n-j}$, where $a_j \geq 0$, $j = g, \ldots, n$ and $\sum_{j=0}^n a_j = 1$.\footnote{Our next result appears in [5].}

**Lemma 4.2.** Suppose that $g > 2n/3$ and that $T \in S(g,n)$. Then $T$ has an eigenvalue of the form $pe^{i\theta}$, where $\theta \in [2\pi/n, 2\pi/g]$, and where $\rho \geq r(\theta)$, where $r(\theta)$ is the (unique) positive solution to the equation $r^g \sin(n\theta) - r^n \sin(g\theta) = \sin((n-g)\theta)$.\footnote{Theorem 3.3.}
Remark 4.1. It is shown in [5] that there is a one-to-one correspondence between the family of complex numbers $r(\theta)e^{it}$, $\theta \in [2\pi/n, 2\pi/g]$, and a family of roots of the polynomial $\lambda^n - \alpha \lambda^{n-g} - (1-\alpha)$, $\alpha \in [0,1]$. Specifically, [5] shows that for each $\alpha \in [0,1]$, there is a $\theta \in [2\pi/n, 2\pi/g]$ such that $r(\theta)e^{it}$ is a root of $\lambda^n - \alpha \lambda^{n-g} - (1-\alpha)$, and conversely that for each $\theta \in [2\pi/n, 2\pi/g]$, there is an $\alpha \in [0,1]$ such that $\lambda^n - \alpha \lambda^{n-g} - (1-\alpha)$ has $r(\theta)e^{it}$ as a root. As $\alpha$ runs from 0 to 1, $\theta$ runs from $2\pi/n$ to $2\pi/g$, while $r(\theta)e^{it}$ interpolates between $e^{2\pi in/n}$ and $e^{2\pi in/g}$.

The following result produces lower bounds on $\lambda_2(g,n)$ for $g > 2n/3$ and for $g \geq 3(n+3)/4$.

Theorem 4.3. a) Suppose that $n \geq 27$ and that $g > 2n/3$. Then $\lambda_2(g,n) \geq (\sqrt{3})^{1/(2n)}$, where $l(n) = \lfloor \frac{n}{2} \rfloor + 1$ if $n \equiv 0, 1 \mod 3$, and $l(n) = \lfloor \frac{n}{2} \rfloor$ if $n \equiv 2 \mod 3$.

b) If $n \geq 3(n+3)/4$, then $\lambda_2(g,n) \geq (\sqrt{3})^{1/(3n)}$.

Proof. a) Let $k = \lfloor \frac{4n}{3} \rfloor$, so that $n = 3k + i$, for some $0 \leq i \leq 2$. Since $g > 2n/3$, it follows that $g \geq 2k + 1$ if $i = 0, 1$, and $g \geq 2k + 2$ if $i = 2$. Let $j_i = 0, j_1 = 1$ and $j_2 = 2$. From Proposition 1.2 b), we find that $\lambda_2(g,n) \geq \lambda_2(2k + j_i, 3k + i)$. From Lemma 4.2 it follows that for each $T \in S(2k + j_i, 3k + i)$, there is a $\theta \in [2\pi/(3k+i), 2\pi/(2k+j_i)]$ such that $\lambda_2(T) \geq r$, where $r$ is the positive solution to the equation $r^{2k+j_i} - r^{3k+i} = \sin((k+i)\theta)$. Evidently for such an $r$ we have $r^{2k+j_i} - r^{3k+i} = \sin((k+i)\theta)$. We utilize this fact in order to establish the desired inequality, it suffices to show that for each $\theta \in [2\pi/(3k+i), 2\pi/(2k+j_i)]$, $\sin((k+i)\theta) \geq \sin((2k+j_i)\theta)$.

To that end, set $t = (k + i - j_i)\theta$, so that $t \in [\frac{\pi}{3} - \frac{2\pi(3j_i-2k)}{3(3k+i)}, \pi - \frac{\pi(3j_i-2k)}{2k+j_i}] \subset [\frac{\pi}{3} - \frac{\pi}{3k+i}, \frac{\pi}{2k+j_i}]$.

Set $b_i = \frac{3j_i-2k}{k+i+j_i}$; we find that $(3j_i)\theta = 3t + b_i t$ and that $(2k+j_i)\theta = 2t + b_i t$. We claim that for each $t \in [\frac{\pi}{3} - \frac{2\pi}{3k+i}, \pi - \frac{\pi}{2k+j_i}]$, $\sin(t) \geq \sin(3t) - \sin(2t + b_i t)$. Let $\cos(t) = x$, so that $-1 < x < 0$. Our claim is equivalent to proving that

\[ 5 - (4x^2 - 2x - 1) \cos(b_i t) \sqrt{1 - x^2} \geq (x - 1)(4x^2 + 2x - 1) \sin(b_i t). \]

From the hypothesis, it follows that $k \geq 9$, so we find that $\sin(b_i t), \cos(b_i t) \geq 0$. First, we note that $-1 < x \leq -\frac{1+\sqrt{5}}{4}$, then we have $4x^2 - 2x - 1 > 4x^2 + 2x - 1 \geq 0$, so that the left side of (4.1) is positive while the right side is nonpositive.

Next, note that if $-\frac{1+\sqrt{5}}{4} < x \leq \frac{1+\sqrt{5}}{4}$, then $4x^2 - 2x - 1 \geq 0 > 4x^2 + 2x - 1$. It then follows that $(5 - (4x^2 - 2x - 1) \cos(b_i t)) \sqrt{1 - x^2} \geq \sqrt{1 - x^2} (6 + 2x - 4x^2) \equiv f(x)$, while $(x - 1)(4x^2 + 2x - 1) \sin(b_i t) \leq (x - 1)(4x^2 + 2x - 1) \equiv g(x)$.

For $-\frac{1+\sqrt{5}}{4} < x \leq \frac{1-\sqrt{5}}{4}$, we find readily that $f(x)$ is an increasing function of $x$, so that in particular, $f(x) \geq \sqrt{\frac{5-\sqrt{5}}{8}} \left( \frac{3-\sqrt{5}}{2} \right) \left( \frac{1+\sqrt{5}}{4} \right) \approx 1.0368312...$ on that interval. A straightforward computation also reveals that $g(x)$ is increasing on the interval $[-\frac{1+\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4}]$, and is maximized on $[-1, 0]$ at $x = \frac{1+\sqrt{5}}{6}$, with $g\left( \frac{1-\sqrt{5}}{6} \right) = \left( \frac{3-\sqrt{5}}{6} \right) \left( \frac{1-\sqrt{5}}{6} \right) + \frac{3}{4} \approx 1.63$. Since $\frac{1-\sqrt{5}}{6} > -0.7$, we find from these considerations that for $-\frac{1+\sqrt{5}}{4} < x \leq -0.7$ we have $g(x) \leq g(-0.7) \approx 0.748 < \frac{1+\sqrt{5}}{4}$.
1.036. On the other hand, if \(-.7 < x \leq \frac{\sqrt{2}}{4}\), then \(f(x) \geq f(-.7) \approx 1.88 > 1.63\). It now follows that for each \(-\frac{\sqrt{2}}{4} \leq x \leq \frac{\sqrt{2}}{4}\), \(f(x) \geq g(x)\).

Finally, if \(-\frac{\sqrt{2}}{4} < x < 0\), the left side of (4.1) is easily seen to exceed \(5\sqrt{1 - (\frac{1 - \sqrt{2}}{4})^2}\), which in turn exceeds the maximum value for \(g(x)\) on \([-1, 0]\). We conclude that (4.1) holds, as desired.

b) Let \(k = \lfloor \frac{n}{4} \rfloor\), so that \(n = 4k + i\) for some \(i = 0, 1, 2, 3\). Since \(g \geq 3(n + 3)/4\), then we have \(g \geq 3k + (9 + 3i)/4\). If \(i = 0\), then \(g \geq 3k\), while if \(i = 1, 2, 3\), then \(g \geq 3k + 3\). Consequently, we have \(\lambda_2(g, n) \geq \lambda_2(3k, 4k)\) if \(i = 0\), and \(\lambda_2(g, n) \geq \lambda_2(3(k + 1), 4(k + 1))\) if \(i = 1, 2, 3\), or equivalently, \(\lambda_2(g, n) \geq \lambda_2(3\lfloor \frac{k}{4} \rfloor, 4\lfloor \frac{k}{4} \rfloor)\).

Set \(j = \lfloor \frac{n}{4} \rfloor\). From Lemma 4.2, we find that \(\lambda_2(3j, 4j) \geq \min\{\sin(4\theta) - \sin(3\theta)\} \geq 3(\frac{1}{4})\) if \(j = 2\), \(2\pi/(3j))\) if \(j = 3\), \(\theta\in[2\pi/(4j), 2\pi/(3j)]\}. We claim that \(\min\{\sin(4\theta) - \sin(3\theta)\} \in [2\pi/(4j), 2\pi/(3j)]\} = (\frac{2\pi}{3j - 1}, \frac{2\pi}{3j})\), from which the result will follow.

To see the claim, let \(x = \cos(j\theta)\) and note that \(x \in [-1/2, 0]\). Further, we have \(\sin(4\theta) - \sin(3\theta) = \sin(\theta)\left(8x^3 - 4x^2 - 4x + 1\right)\). Consequently, \(\min\{\sin(4\theta) - \sin(3\theta)\} \in [2\pi/(4j), 2\pi/(3j)]\} = \min\{\frac{1}{8x^3 - 4x^2 - 4x + 1}\}x \in [-1/2, 0]\}. The claim now follows from a standard calculus computation.

Remark 4.2. Note that \(\sqrt[3]{2} - 1 \approx 0.6130718\ldots\).

Remark 4.3. We note that Theorem 4.3 provides an estimate on \(r(\theta)\) for the case that \(g > 2n/3\); that estimate is a clear improvement on that of [6], which proves a lower bound of \(\frac{\sin(\pi(n - 1))}{n}\) on that quantity.

Our final result considers the case that \(n \to \infty\), while \(n - g\) is fixed. In the proof, we use the notation \(O(\frac{1}{n^2})\) to denote a sequence \(s_n\) with the property that \(n^k s_n\) is a bounded sequence.

Theorem 4.4. Suppose that \(i \geq 1\) is fixed. Then \(\lambda_2(n - i, n) \geq 1 - \frac{\pi^2}{2n^2} + O(\frac{1}{n^3})\).

Proof. From Lemma 4.2, we find that for \(n > 3i\) we have

\[
\lambda_2(n - i, n) \geq \left(\min \left\{\frac{\sin(\theta)}{\sin(n\theta) - \sin((n - i)\theta)}\right\} \theta \in [2\pi/n, 2\pi/(n - i)]\right)^{\frac{1}{i}}.
\]

Let \(\theta_0\) be a critical point of the function \(\frac{\sin(\theta)}{\sin(n\theta) - \sin((n - i)\theta)}\) on the interval \([2\pi/n, 2\pi/(n - i)]\). Then we have

\[
\sin(i\theta_0)(n \cos(n\theta_0) - (n - i) \cos((n - i)\theta_0)) = i \cos(i\theta_0)(\sin(n\theta_0) - \sin((n - i)\theta_0)).
\]

Let \(\theta_0 = \frac{\alpha n}{n} + \frac{\pi}{n}\) where \(a = O(1)\). We then have \(n\theta_0 = 2\pi + \frac{\alpha n}{n}\), \((n - i)\theta_0 = 2\pi - \left(\frac{2\pi - \alpha n}{n} + \frac{\alpha n}{n}\right)\) and \(i\theta_0 = \frac{2\pi}{n} + \frac{\pi}{n}\). Expanding the equation above for \(\theta_0\) to terms in \(O(\frac{1}{n^2})\), we have \(\left(\frac{\alpha n}{n} + \frac{\pi}{n}\right) \left(n \left(1 - \frac{\alpha n}{2n^2}\right) - (n - i) \left(1 - \frac{(2\pi - \alpha n)}{2n^2}\right)\right) = i \left(1 - \frac{\pi^2}{2n^2}\right) \left[\frac{\alpha n}{n} + \frac{(2\pi - \alpha n)}{n} + \frac{\pi}{n}\right] + O(\frac{1}{n^2})\). Collecting terms and simplifying eventually yields \(\frac{(2\pi - \alpha n)^2 - \alpha^2}{2n^2} = O(\frac{1}{n})\) from which we conclude that \(a = i + O(\frac{1}{n})\).

Next, we write \(\theta_0 = \frac{2\pi}{n} + \frac{\pi}{n^2} + \frac{b}{n^2}\), where \(b = O(1)\). As above, we find that \(n\theta_0 = 2\pi + \frac{\pi}{n^2} + \frac{b}{n^2}\), \((n - i)\theta_0 = 2\pi - \left(\frac{\pi}{n^2} + \frac{(i - b)\pi}{n^2} + \frac{b}{n^2}\right)\) and \(i\theta_0 = \frac{2\pi}{n} + \frac{\pi^2}{n^2} + \frac{ab}{n^2}\).
Girth and Subdominant Eigenvalues

From this it follows that

$$\frac{\sin(i\theta_0)}{\sin((n-i)\theta_0)} = \frac{2\pi n + 2i\alpha + \pi i}{2\pi n + 2i\alpha - \pi i} = 1 - \frac{\pi^2 i^2}{2n^2} + O\left(\frac{1}{n^3}\right).$$

Thus we have $\lambda_2(n-i,n) \geq \left(1 - \frac{\pi^2 i^2}{2n^2} + O\left(\frac{1}{n^3}\right)\right)^{\frac{1}{n-1}} = 1 - \frac{\pi^2 i^2}{2n^2} + O\left(\frac{1}{n^3}\right)$, as desired. \(\square\)

**Remark 4.4.** Suppose that we have a matrix $T \in S(n-1,n)$. Then the characteristic polynomial of $T$ is given by $p_\alpha(\lambda) \equiv \lambda^3 - \alpha \lambda - (1 - \alpha)$, for some $\alpha \in [0,1]$. Conversely, for each $\alpha \in [0,1]$, there is a matrix $T \in S(n-1,n)$ whose characteristic polynomial is $p_\alpha$, namely the companion matrix of that polynomial. Thus we see that the eigenvalues of matrices in $S(n-1,n)$ are in one-to-one correspondence with the roots of polynomials of the form $p_\alpha, \alpha \in [0,1]$. For such a polynomial, we say that a root $\lambda$ is a subdominant root if $\lambda \neq 1$ and $\lambda$ has maximum modulus among the roots of the polynomial that are distinct from 1. In particular, we find that discussing the subdominant roots of the polynomials $p_\alpha, \alpha \in [0,1]$ is equivalent to discussing the subdominant eigenvalues of the matrices in $S(n-1,n)$.

Fix a value of $n \geq 4$. It follows from Corollary 2.1 of [5] that for each $\alpha \in [0,1]$, there is precisely one root of $p_\alpha$ whose argument lies in $[2\pi/n, 2\pi/(n-1)]$ (including multiplicities). Denote that root by $\sigma(\alpha)$. Evidently an analogous statement holds for the interval $[2\pi - 2\pi/n(n-1), 2\pi - 2\pi/n]$, and we claim that in fact $\sigma(\alpha)$ and $\sigma'(\alpha)$ are subdominant roots for $p_\alpha$.

To see the claim, first suppose that $\alpha \in (0,1)$, and that $z_1$ and $z_2$ are two roots of $p_\alpha$ of equal moduli. Writing $z_1 = r e^{i\theta_1}, z_2 = r e^{i\theta_2}$, and substituting each into the equation $p_\alpha(\lambda) = 0$, we find that $r^2 = |\alpha| e^{i\theta_1} + 1 - \alpha|^2 = |\alpha| e^{i\theta_2} + 1 - \alpha|^2$. It follows that $\alpha^2 r^2 + (1 - \alpha)^2 + 2\alpha(1 - \alpha) \rho \cos(\theta_1) = \alpha^2 r^2 + (1 - \alpha)^2 + 2\alpha(1 - \alpha) \rho \cos(\theta_2)$, from which we conclude that $\cos(\theta_1) = \cos(\theta_2)$. Consequently, we find that for each $\alpha \in (0,1)$, if $z_1$ and $z_2$ are roots of $p_\alpha$ that have equal moduli, then either $z_1 = z_2$ or $z_1 = \overline{z_2}$.

For each $\alpha \in [0,1]$, denote the roots of $p_\alpha$ that are distinct from 1 and whose argument fall outside of $[2\pi/n, 2\pi/(n-1)] \cup [2\pi - 2\pi/(n-1), 2\pi - 2\pi/n]$ by $\gamma_1(\alpha), \ldots, \gamma_{n-3}(\alpha)$, labeled in nondecreasing order according to their arguments. Suppose that $\exists \alpha_1, \alpha_2 \in (0,1)$ such that $|\sigma(\alpha_1)| > \max\{|\gamma_i(\alpha_1)|; i = 1, \ldots, n-3\}$ and $|\sigma(\alpha_2)| < \max\{|\gamma_i(\alpha_2)|; i = 1, \ldots, n-3\}$ from the continuity of the roots of $p_\alpha$ in the parameter $\alpha$, and the intermediate value theorem, we find that $\exists \alpha_3 \in (0,1)$ such that $|\sigma(\alpha_3)| = \max\{|\gamma_i(\alpha_3)|; i = 1, \ldots, n-3\}$. Hence for some $i$ we have either $\gamma_i(\alpha_3) = \sigma(\alpha_3)$ or $\gamma_i(\alpha_3) = \overline{\sigma(\alpha_3)}$, a contradiction since the argument of $\gamma_i$ falls outside of $[2\pi/n, 2\pi/(n-1)]\cup[2\pi - 2\pi/(n-1), 2\pi - 2\pi/n]$. Consequently, we find that one of the following alternatives must hold: either $|\sigma(\alpha)| > \max\{|\gamma_i(\alpha)|; i = 1, \ldots, n-3\}$ for all $\alpha \in (0,1)$, or $|\sigma(\alpha)| < \max\{|\gamma_i(\alpha)|; i = 1, \ldots, n-3\}$ for all $\alpha \in (0,1)$.

Next, we claim that for all sufficiently small $\alpha > 0, \sigma(\alpha)$ is a subdominant eigenvalue of $p_\alpha$. To see this, observe that at $\alpha = 0$, the roots of $p_\alpha$ that are distinct from 1 are given by $e^{2\pi j/n}, 1 \leq j \leq n-1$. Note that these roots are distinct, there is a neighbourhood of $\alpha = 0$ on which each root of $p_\alpha$ is a differentiable function of $\alpha$.

Fix an index $l$ such that either $1 \leq l < (n-2)/2$ or $(n-2)/2 < l \leq n-3$ and
consider $\gamma_l(\alpha)$. We write $\gamma_l(\alpha) = \rho e^{i\theta}$, where on the right hand side, the explicit dependence on $\alpha$ is suppressed. Considering the real and imaginary parts of the equation $p_\alpha(\rho e^{i\theta}) = 0$, we find that for each $0 < \alpha \leq 1$ we have
\begin{equation}
\rho^n \cos(n\theta) - 1 = \alpha (\rho \cos(\theta) - 1) \tag{4.3}
\end{equation}
and
\begin{equation}
\rho^{n-1} \sin(n\theta) = \alpha \sin(\theta). \tag{4.4}
\end{equation}
In particular, crossmultiplying (4.3) and (4.4), canceling the common factor of $\alpha$, and simplifying, we find that for each $0 < \alpha \leq 1$, we have
\begin{equation}
\rho^{n-1} \sin(n\theta) - \rho^n \sin((n-1)\theta) = \sin(\theta). \tag{4.5}
\end{equation}
(Observe that in fact (4.5) also holds when $\alpha = 0$, since then $\rho = 1$ and $\theta = 2\pi (l+1)/n$.) Differentiating (4.4) with respect to $\alpha$ and evaluating at $\alpha = 0$, it follows that $\frac{d\rho}{d\alpha}|_{\alpha=0} = \frac{\sin(2\pi (l+1)/n)}{n}$. Differentiating (4.5) with respect to $\alpha$ (via the chain rule) and evaluating at $\alpha = 0$ then yields $\frac{d\rho}{d\alpha}|_{\alpha=0} = -\frac{1-\cos(2\pi (l+1)/n)}{n}$. Similar arguments show that if $l = (n-2)/2$, then $\frac{d\rho}{d\alpha}|_{\alpha=0} = \frac{1}{n}$, and that $\frac{d\sigma}{d\alpha}|_{\alpha=0} = \frac{1-\cos(2\pi/n)}{n}$.

We conclude that for all sufficiently small $\alpha > 0$, $|\sigma(\alpha)| = 1 - \alpha \left(\frac{1-\cos(2\pi/n)}{n}\right) + O(\alpha^2) > 1 - \alpha \left(\frac{1-\cos(2\pi (l+1)/n)}{n}\right) + O(\alpha^2) = |\gamma_l(\alpha)|, l = 1, \ldots, n-3$. Hence, for such $\alpha$, $\sigma$ (and $\overline{\sigma}$) are subdominant roots of $p_\alpha$. From the considerations above, we conclude that for each $\alpha \in [0,1]$, $\sigma(\alpha)$ is a subdominant root of $p_\alpha$, as claimed.

From the claim, it now follows that $\lambda_2(n-1,n) = \min\{||\sigma(\alpha)||: \alpha \in [0,1]\} = \min\{|\sigma(\theta)|: \theta \in [2\pi/n,2\pi/(n-1)]\}$. Arguing as in Theorem 4.4, there is a $\theta_0 \in [2\pi/n,2\pi/(n-1)]$ such that $\frac{\sin(\theta_0)}{\sin(\theta_0) - \sin((n-1)\theta_0)} = 1 - \frac{\pi^2}{2n^2} + O\left(\frac{1}{n}\right)$, which yields $\left(1 - \frac{\pi^2}{2n^2} + O\left(\frac{1}{n}\right)\right)^{1/n} \geq \sin(\theta_0) \geq \lambda_2(n-1,n)$. Applying Theorem 4.4, we find that $\left(1 - \frac{\pi^2}{2n^2} + O\left(\frac{1}{n}\right)\right)^{1/n} \geq \lambda_2(n-1,n) \geq \left(1 - \frac{\pi^2}{2n^2} + O\left(\frac{1}{n}\right)\right)^{1/(n-1)}$. But since both the upper and lower bounds on $\lambda_2(n-1,n)$ can be written as $1 - \frac{\pi^2}{2n^2} + O\left(\frac{1}{n^3}\right)$, we conclude that $\lambda_2(n-1,n) = 1 - \frac{\pi^2}{2n^2} + O\left(\frac{1}{n^3}\right)$.

Acknowledgment. The author is grateful to an anonymous referee, whose careful reading and constructive comments led to a number of improvements to this paper.

REFERENCES


