SUBDIRECT SUMS OF NONSINGULAR $M$-MATRICES AND OF THEIR INVERSES∗

RAFAEL BRU†, FRANCISCO PEDROCHE†, AND DANIEL B. SZYLD‡

Abstract. The question of when the subdirect sum of two nonsingular $M$-matrices is a nonsingular $M$-matrix is studied. Sufficient conditions are given. The case of inverses of $M$-matrices is also studied. In particular, it is shown that the subdirect sum of overlapping principal submatrices of a nonsingular $M$-matrix is a nonsingular $M$-matrix. Some examples illustrating the conditions presented are also given.

AMS subject classifications. 15A48.

Key words. Subdirect sum, $M$-matrices, Inverse of $M$-matrix, Overlapping blocks.

1. Introduction. Subdirect sum of matrices are generalizations of the usual sum of matrices (a $k$-subdirect sum is formally defined below in Section 2). They were introduced by Fallat and Johnson in [3], where many of their properties were analyzed. For example, they showed that the subdirect sum of positive definite matrices, or of symmetric $M$-matrices, are positive definite or symmetric $M$-matrices, respectively. They also showed that this is not the case for $M$-matrices: the sum of two $M$-matrices may not be an $M$-matrix. One goal of the present paper is to give sufficient conditions so that the subdirect sum of nonsingular $M$-matrices is a nonsingular $M$-matrix. We also treat the case of the subdirect sum of inverses of $M$-matrices.

Subdirect sums of two overlapping principal submatrices of a nonsingular $M$-matrix appear naturally when analyzing additive Schwarz methods for Markov chains or other matrices [2], [4]. In this paper we show that the subdirect sum of two overlapping principal submatrices of a nonsingular $M$-matrix is a nonsingular $M$-matrix.

The paper is structured as follows. In Section 2 we focus on the nonsingularity of the subdirect sum of any pair of nonsingular matrices, giving an explicit expression for the inverse. In Section 2.1 we study the $k$-subdirect sum of two nonsingular $M$-matrices and in particular, the case of subdirect sums of overlapping blocks of nonsingular $M$-matrices. In Section 2.3 we extend some results to the subdirect sum of more than two nonsingular $M$-matrices. In Section 3 we analyze the subdirect sum of two inverses. Finally, in Section 4 we mention some open questions on subdirect sums of $P$-matrices. Throughout the paper we give examples which help illustrate the theoretical results.

∗Received by the editors 16 February 2005. Accepted for publication 22 June 2005. Handling Editor: Michael Neumann.

†Institut de Matemàtica Multidisciplinar, Universitat Politècnica de València, Camí de Vera s/n, 46022 València, Spain (rbru@mat.upv.es, pedroche@mat.upv.es). Supported by Spanish DGI and FEDER grant MTM2004-02988 and by the Oficina de Ciencia y Tecnología de la Presidencia de la Generalitat Valenciana under project GRUPOS03/002.

‡Department of Mathematics, Temple University, Philadelphia, PA 19122-6094, U.S.A. (szyl@math.temple.edu). Supported in part by the U.S. National Science Foundation under grant DMS-0207525.
2. Subdirect sums of nonsingular matrices. Let $A$ and $B$ be two square matrices of order $n_1$ and $n_2$, respectively, and let $k$ be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let $A$ and $B$ be partitioned into $2 \times 2$ blocks as follows:

$$
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix},
$$

(2.1)

where $A_{22}$ and $B_{11}$ are square matrices of order $k$. Following [3], we call the following square matrix of order $n = n_1 + n_2 - k$,

$$
C = \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{21} & A_{22} + B_{11} & B_{12} \\
0 & B_{21} & B_{22}
\end{bmatrix}
$$

(2.2)

the $k$-subdirect sum of $A$ and $B$ and denote it by $C = A \oplus_k B$.

We are interested in the case when $A$ and $B$ are nonsingular matrices. We partition the inverses of $A$ and $B$ conformably to (2.1) and denote its blocks as follows:

$$
A^{-1} = \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
\hat{A}_{21} & \hat{A}_{22}
\end{bmatrix}, \quad B^{-1} = \begin{bmatrix}
\hat{B}_{11} & \hat{B}_{12} \\
\hat{B}_{21} & \hat{B}_{22}
\end{bmatrix},
$$

(2.3)

where, as before, $\hat{A}_{22}$ and $\hat{B}_{11}$ are square of order $k$.

In the following result we show that nonsingularity of matrix $\hat{A}_{22} + \hat{B}_{11}$ is a necessary and sufficient condition for the $k$-subdirect sum $C$ to be nonsingular. The proof is based on the use of the relation $n = n_1 + n_2 - k$ to properly partition the indicated matrices.

**Theorem 2.1.** Let $A$ and $B$ be nonsingular matrices of order $n_1$ and $n_2$, respectively, and let $k$ be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let $A$ and $B$ be partitioned as in (2.1) and their inverses be partitioned as in (2.3). Let $C = A \oplus_k B$. Then $C$ is nonsingular if and only if $C = \hat{A}_{22} + \hat{B}_{11}$ is nonsingular.

**Proof.** Let $I_m$ be the identity matrix of order $m$. The theorem follows from the following relation:

$$
\begin{bmatrix}
A^{-1} & O \\
O & I_{n-n_1}
\end{bmatrix}
C
\begin{bmatrix}
I_{n-n_2} & O \\
O & B^{-1}
\end{bmatrix}
= \begin{bmatrix}
I_{n-n_2} & \hat{A}_{12} \\
O & B
\end{bmatrix}
\begin{bmatrix}
\hat{A}_{12} & O \\
O & I_{n-n_1}
\end{bmatrix}.
$$

(2.4)

2.1. Nonsingular $M$-matrices. Given $A = \{a_{ij}\} \in \mathbb{R}^{m \times n}$, we write $A > O$ ($A \geq O$) to indicate $a_{ij} > 0$ ($a_{ij} \geq 0$), for $i = 1, \ldots, m$, $j = 1, \ldots, n$, and such matrices are called positive (nonnegative). Similarly, $A \geq B$ when $A - B \geq O$. Square matrices which have nonpositive off-diagonal entries are called $Z$-matrices. We call a $Z$-matrix $M$ a nonsingular $M$-matrix if $M^{-1} \geq O$. We recall some properties of these matrices; see [1], [8]:

(i) The diagonal of a nonsingular $M$-matrix is positive.

(ii) If $B$ is a $Z$-matrix and $M$ is a nonsingular $M$-matrix, and $M \leq B$, then $B$ is also a nonsingular $M$-matrix. In particular, any matrix obtained from a nonsingular $M$-matrix by setting certain off-diagonal entries to zero is also a nonsingular $M$-matrix.
(iii) A matrix $M$ is a nonsingular $M$-matrix if and only if each principal submatrix of $M$ is a nonsingular $M$-matrix.

(iv) A $Z$-matrix $M$ is a nonsingular $M$-matrix if and only if there exists a positive vector $x > 0$ such that $Mx > 0$.

We first consider the $k$-subdirect sum of nonsingular $Z$-matrices. From (2.4) we can explicitly write

$$C^{-1} = \begin{bmatrix} I_{n-n_2} & 0 & 0 \\ O & B^{-1} \end{bmatrix} \begin{bmatrix} I_{n-n_2} & -\hat{A}_{12} \hat{H}^{-1} \hat{B}_{12} \\ O & \hat{H}^{-1} \end{bmatrix} \begin{bmatrix} -\hat{A}_{12} \hat{H}^{-1} \hat{B}_{12} \\ O \\ I_{n-n_1} \end{bmatrix} \begin{bmatrix} A^{-1} & O \\ O & I_{n-n_1} \end{bmatrix}$$

from which we obtain

$$C^{-1} = \begin{bmatrix} \hat{A}_{11} - \hat{A}_{12} \hat{H}^{-1} \hat{A}_{21} & \hat{A}_{12} - \hat{A}_{12} \hat{H}^{-1} \hat{A}_{22} & \hat{A}_{12} \hat{H}^{-1} \hat{B}_{12} \\ \hat{B}_{11} \hat{H}^{-1} \hat{A}_{21} & \hat{B}_{11} \hat{H}^{-1} \hat{A}_{22} - \hat{B}_{11} \hat{H}^{-1} \hat{B}_{12} + \hat{B}_{12} \\ \hat{B}_{21} \hat{H}^{-1} \hat{A}_{21} & \hat{B}_{21} \hat{H}^{-1} \hat{A}_{22} - \hat{B}_{21} \hat{H}^{-1} \hat{B}_{12} + \hat{B}_{22} \end{bmatrix}$$

(2.5)

and therefore we can state the following immediate result.

**Theorem 2.2.** Let $A$ and $B$ be nonsingular $Z$-matrices of order $n_1$ and $n_2$, respectively, and let $k$ be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let $A$ and $B$ be partitioned as in (2.1) and their inverses be partitioned as in (2.3). Let $C = A \oplus_k B$. Let $H = A_{22} + B_{11}$ be nonsingular. Then $C$ is a nonsingular $M$-matrix if and only if each of the nine blocks of $C^{-1}$ in (2.5) is nonnegative.

We consider now the case where $A$ and $B$ are nonsingular $M$-matrices. It was shown in [3] that even if $H = A_{22} + B_{11}$ is a nonsingular $M$-matrix, this does not guarantee that $C = A \oplus_k B$ is a nonsingular $M$-matrix. We point out that this matrix $H$ is not the matrix obtained from $A^{-1}$ and $B^{-1}$ and used in Theorem 2.1. The fact that $H$ is a nonsingular $M$-matrix is a necessary but not a sufficient condition for $C$ to be a nonsingular $M$-matrix. Sufficient conditions are presented in the following result.

**Theorem 2.3.** Let $A$ and $B$ be nonsingular $M$-matrices partitioned as in (2.1). Let $x_1 > 0 \in \mathbb{R}^{(n_1-k) \times 1}$, $y_1 > 0 \in \mathbb{R}^{k \times 1}$, $x_2 > 0 \in \mathbb{R}^{k \times 1}$ and $y_2 > 0 \in \mathbb{R}^{(n_2-k) \times 1}$ be such that

$$A \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} > 0, \quad B \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} > 0.$$  

(2.6)

Let $H = A_{22} + B_{11}$ be a nonsingular $M$-matrix and let

$$y = H^{-1} (A_{22} y_1 + B_{11} x_2).$$

(2.7)

Then if $y \leq y_1$ and $y \leq y_2$ the $k$-subdirect sum $C = A \oplus_k B$ is a nonsingular $M$-matrix.

*Proof.* We will show that there exists $u > 0$ such that $Cu > 0$. We first note that from (2.6) we get

$$\begin{cases} A_{11} x_1 + A_{12} y_1 > 0 \\ A_{21} x_1 + A_{22} y_1 > 0 \end{cases}, \quad \begin{cases} B_{11} x_2 + B_{12} y_2 > 0 \\ B_{21} x_2 + B_{22} y_2 > 0 \end{cases}.$$  

(2.8)
Subdirect Sums of Nonsingular $M$-matrices and of Their Inverses

Taking $u = \begin{bmatrix} x_1 \\ y \\ y_2 \end{bmatrix}$ and partitioning $C$ as in (2.2) we obtain
\[
Cu = \begin{bmatrix}
A_{11}x_1 + A_{12}y \\
A_{21}x_1 + (A_{22} + B_{11})y + B_{12}y_2 \\
B_{21}y + B_{22}y_2
\end{bmatrix}.
\] (2.9)

Since $A_{21} \leq O$ and $B_{12} \leq O$, from (2.8) it follows that $A_{22}y_1 > 0$ and $B_{11}x_2 > 0$. Since $H^{-1} \geq O$, from (2.7) we have that $y$ is positive, and consequently, so is $u$, i.e., $u > 0$. We will show that $Cu > 0$ one block of rows in (2.9) at a time. If $y \leq y_1$, as $A_{12} \leq 0$, we have that $A_{12}y \geq A_{12}y_1$ and again using (2.8) we obtain that the first block of rows of $Cu$ is positive. In a similar way, the condition $y \leq x_2$ together with the last equation of (2.8) allows to conclude that the third block of rows of $Cu$ is positive. Finally, substituting $y$ given by (2.7) in the second row of $Cu$ and considering (2.8) we conclude that the second block of rows of $Cu$ is also positive. ∎

Note that $A$ and $B$ are nonsingular $M$-matrices and therefore the positive vectors $(x_1, y_1)$ and $(x_2, y_2)$ of (2.6) always exist. This theorem gives sufficient but not necessary conditions for $C = A \oplus_k B$ to be a nonsingular $M$-matrix, as illustrated in Example 2.5 further below.

**Example 2.4.** The matrices
\[
A = \begin{bmatrix}
3 & 1/2 & -1 \\
-1/2 & 2 & -3 \\
-1 & -1 & 4
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
1 & -2 & -1/3 \\
-3 & 9 & 0 \\
-2 & -1/2 & 6
\end{bmatrix},
\]

and the vectors
\[
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 2 \\ 1 \end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \\ 1 \end{bmatrix}
\]

satisfy the inequalities (2.6), and computing the vector $y$ from (2.7) we get $y \approx (1.95, 0.87)^T$, which satisfy $y \leq y_1$ and $y \leq x_2$. Therefore the 2-subdirect sum
\[
C = \begin{bmatrix}
3 & -2 & -1 & 0 \\
-1/2 & 3 & -5 & -1/3 \\
-1 & -4 & 13 & 0 \\
0 & -2 & -1/2 & 6
\end{bmatrix}
\]
is a nonsingular $M$-matrix in accordance with Theorem 2.3.

**Example 2.5.** The matrices
\[
A = \begin{bmatrix}
5 & -1/2 & -1/3 \\
-1 & 4 & -2 \\
-1 & -6 & 10
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
1 & -2 & -1/3 \\
-3 & 9 & 0 \\
-2 & -1/2 & 6
\end{bmatrix},
\]

and the vectors
\[
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \\ 1 \end{bmatrix}.
satisfy the inequalities (2.6), but computing vector \( y \) from (2.7) we obtain
\[ y \approx (1.18, 0.85)^T, \]
which does not satisfy the conditions of Theorem 2.3. Nevertheless the 2-subdirect sum
\[
C = A \oplus_2 B = \begin{bmatrix}
5 & -1/2 & -1/3 & 0 \\
-1 & 5 & -4 & -1/3 \\
-1 & -9 & 19 & 0 \\
0 & -2 & -1/2 & 6
\end{bmatrix}
\]
is a nonsingular \( M \)-matrix.

In the special case of \( A \) and \( B \) block lower and upper triangular nonsingular \( M \)-matrices, respectively, the results of Theorems 2.2 and 2.3 are easy to establish. Let
\[
A = \begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_{11} & B_{12} \\
0 & B_{22}
\end{bmatrix},
\]
(2.10)
with \( A_{22} \) and \( B_{11} \) square matrices of order \( k \).

**Theorem 2.6.** Let \( A \) and \( B \) be nonsingular lower and upper block triangular nonsingular \( M \)-matrices, respectively, partitioned as in (2.10). Then \( C = A \oplus_k B \) is a nonsingular \( M \)-matrix.

**Proof.** We can repeat the same argument as in the proof of Theorem 2.3 with the advantage of having \( A_{12} = O \) and \( B_{21} = O \). Note that conditions \( y \leq y_1 \) and \( y \leq x_2 \) are not necessary here because the first and last block of rows of \( C \) in (2.9) are automatically positive in this case. \( \Box \)

**Remark 2.7.** The expression of \( C^{-1} \) is given by (2.5). In this particular case of block triangular matrices we have \( A_{12} = O \), \( B_{21} = O \), \( A_{22} = A_{22}^{-1} \), \( B_{11} = B_{11}^{-1} \), from which \( C = A_{22}^{-1} + B_{11}^{-1} \). If, in addition, \( A_{22} = B_{11} \), then we obtain
\[
C^{-1} = \begin{bmatrix}
A_{11}^{-1} & O & O \\
-\frac{1}{2}A_{22}^{-1}A_{21}A_{11}^{-1} & \frac{1}{2}A_{22}^{-1} & -\frac{1}{2}A_{22}^{-1}B_{12}B_{22}^{-1}
\end{bmatrix} \geq O.
\]

**Example 2.8.** The matrices
\[
A = \begin{bmatrix}
3 & 0 & 0 \\
-1 & 5 & -1 \\
-1 & -9 & 5
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
6 & -2 & -1 \\
-4 & 3 & -3 \\
0 & 0 & 2
\end{bmatrix}
\]
satisfy the hypotheses of Theorem 2.6. The matrices \( C = A \oplus_2 B \) and \( C^{-1} \) are
\[
C = \begin{bmatrix}
3 & 0 & 0 & 0 \\
-1 & 11 & -3 & -1 \\
-1 & -13 & 8 & -3 \\
0 & 0 & 0 & 2
\end{bmatrix}, \quad C^{-1} = \begin{bmatrix}
1/3 & 0 & 0 & 0 \\
11/147 & 8/49 & 3/49 & 17/98 \\
8/49 & 13/49 & 11/49 & 23/49 \\
0 & 0 & 0 & 1/2
\end{bmatrix}
\]
and therefore \( C \) is a nonsingular \( M \)-matrix as expected.
Subdirect Sums of Nonsingular $M$-matrices and of Their Inverses

In some applications, such as in domain decomposition [6], [7], matrices $A$ and $B$ partitioned as in (2.1) arise with a common block, i.e., $A_{22} = B_{11}$. In the next example we show that even if $A$ and $B$ are nonsingular $M$-matrices, and so is the common block, we cannot ensure that $C = A \oplus_k B$ is a nonsingular $M$-matrix.

**Example 2.9.** The matrices

\[
A = \begin{bmatrix}
370 & -342 & -318 \\
-448 & 737 & -107 \\
-46 & -190 & 444
\end{bmatrix}, \quad B = \begin{bmatrix}
737 & -107 & -134 \\
-190 & 444 & -440 \\
-885 & -182 & 603
\end{bmatrix}
\]

are nonsingular $M$-matrices with $A_{22} = B_{11}$ an $M$-matrix, but $C = A \oplus_2 B$ is not an $M$-matrix, since we have

\[
C = \begin{bmatrix}
370 & -342 & -318 & 0 \\
-448 & 1474 & -214 & -134 \\
-46 & -380 & 888 & -440 \\
0 & -885 & -182 & 603
\end{bmatrix}
\]

and

\[
C^{-1} \approx \begin{bmatrix}
-0.0291 & -0.0242 & -0.0204 & -0.0203 \\
-0.0145 & -0.0109 & -0.0098 & -0.0096 \\
-0.0214 & -0.0163 & -0.0132 & -0.0133 \\
-0.0277 & -0.0210 & -0.0183 & -0.0164
\end{bmatrix}.
\]

In the next section we shall see that when $A$ and $B$ share a block and they are submatrices of a given nonsingular $M$-matrix, the resulting $k$-subdirect sum is in fact a nonsingular $M$-matrix.

2.2. **Overlapping $M$-matrices.** In this section we restrict $A$ and $B$ to be principal submatrices of a given nonsingular $M$-matrix and such that they have a common block. Let

\[
M = \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
\]

be a nonsingular $M$-matrix with $M_{22}$ square matrix of order $k \geq 1$ and let

\[
A = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
M_{22} & M_{23} \\
M_{32} & M_{33}
\end{bmatrix}
\]

be of order $n_1$ and $n_2$, respectively. The $k$-subdirect sum of $A$ and $B$ is thus given by

\[
C = A \oplus_k B = \begin{bmatrix}
M_{11} & M_{12} & O \\
M_{21} & 2M_{22} & M_{23} \\
O & M_{32} & M_{33}
\end{bmatrix}.
\]

In the following theorem we show that $C$ is a nonsingular $M$-matrix.
Theorem 2.10. Let $M$ be a nonsingular $M$-matrix partitioned as in (2.11), and let $A$ and $B$ be two overlapping principal submatrices given by (2.12). Then the \( k \)-subdirect sum $C = A \oplus_k B$ is a nonsingular $M$-matrix.

Proof. Let us construct an $n \times n$ Z-matrix $T$ as follows:

$$
T = \begin{bmatrix}
M_{11} & 2M_{12} & M_{13} \\
M_{21} & 2M_{22} & M_{23} \\
M_{31} & 2M_{32} & M_{33}
\end{bmatrix}.
$$

(2.14)

Then $T = M \text{diag}(I, 2I, I)$ and we get $T^{-1} = \text{diag}(I, (1/2)I, I)M^{-1} \geq 0$. Then $T$ is a nonsingular $M$-matrix. Finally since $C$ is a Z-matrix and $C \geq T$ we conclude that $C$ is a nonsingular $M$-matrix.

Example 2.11. The following nonsingular $M$-matrix is partitioned as in (2.11):

$$
M = \begin{bmatrix}
13/14 & -1/42 & -1/42 & -9/186 & -3/46 \\
-3/7 & 21/23 & -1/5 & -1/21 & -1/93 & -6/23 \\
-1/7 & -7/46 & 17/20 & -1/14 & -1/186 & -2/23 \\
\end{bmatrix}.
$$

(2.15)

Taking overlapping submatrices $A$ and $B$ as in (2.12) the 3-subdirect sum $C = A \oplus_3 B$ is given by

$$
C = \begin{bmatrix}
13/14 & -1/42 & -1/42 & -9/186 & 0 \\
-3/7 & 21/23 & -1/5 & -1/21 & -1/93 & 0 \\
-1/7 & -7/46 & 17/10 & -1/7 & -1/93 & -2/23 \\
0 & 0 & -2/15 & -2/7 & -7/62 & 83/92
\end{bmatrix}
$$

and it is a nonsingular $M$-matrix according to Theorem 2.10. In fact, we have that

$$
C^{-1} \approx \begin{bmatrix}
1.3500 & 0.3977 & 0.2624 & 0.1609 & 0.2103 & 0.1232 \\
0.7628 & 1.4108 & 0.3383 & 0.2085 & 0.2185 & 0.1476 \\
0.3007 & 0.2845 & 0.7422 & 0.2006 & 0.1824 & 0.1763 \\
1.1024 & 1.1571 & 0.8927 & 1.6092 & 1.3118 & 0.8940 \\
0.4854 & 0.5256 & 0.5116 & 0.4379 & 0.9664 & 0.4013 \\
0.4543 & 0.4743 & 0.4564 & 0.5941 & 0.5634 & 1.4679
\end{bmatrix}.
$$

2.3. $k$-subdirect sum of $p$ $M$-matrices. In this section we extend Theorems 2.3 and 2.10 to the subdirect sum of several nonsingular $M$-matrices. Example 2.14 later in the section illustrates the notation used in the proofs.

Theorem 2.12. Let $A_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \ldots, p$, be nonsingular $M$-matrices partitioned as

$$
A_i = \begin{bmatrix}
A_{i,11} & A_{i,12} \\
A_{i,21} & A_{i,22}
\end{bmatrix}
$$

(2.16)
Subdirect Sums of Nonsingular $M$-matrices and of Their Inverses

with $A_{i,11}$ a square matrix of order $k_{i-1} \geq 1$ and $A_{i,22}$ a square matrix of order $k_i \geq 1$, i.e., $n_i = k_{i-1} + k_i$. Since $A_i$ are nonsingular $M$-matrices we have that there exist $x_i > 0 \in \mathbb{R}^{(n_i-k_i) \times 1}$ and $y_i > 0 \in \mathbb{R}^{k_i \times 1}$ such that

$$
A_i \begin{bmatrix} x_i \\ y_i \end{bmatrix} > 0, \quad i = 1, \ldots, p.
$$

(2.17)

Let $C_0 = A_1$ and define the following $p - 1$ $k_i$-subdirect sums:

$$
C_i = C_{i-1} \oplus_k A_{i+1}, \quad i = 1, \ldots, p - 1,
$$

i.e.,

$$
egin{align*}
C_1 &= A_1 \oplus_{k_1} A_2, \\
C_2 &= (A_1 \oplus_{k_2} A_2) \oplus_{k_3} A_3 = C_1 \oplus_{k_3} A_3, \\
& \vdots \\
C_{p-1} &= (A_1 \oplus_{k_2} A_2 \oplus_{k_3} \cdots \oplus_{k_{p-2}} A_{p-1}) \oplus_{k_{p-1}} A_p = C_{p-2} \oplus_{k_{p-1}} A_p.
\end{align*}
$$

Each subdirect sum $C_i$ is of order $m_i$, such that $m_0 = n_1$ and

$$
m_i = m_{i-1} + n_{i+1} - k_i = m_{i-1} + k_{i+1}, \quad i = 1, \ldots, p - 1.
$$

Let us partition $C_i$ in the form

$$
C_i = \begin{bmatrix} C_{i,11} & C_{i,12} \\ C_{i,21} & C_{i,22} \end{bmatrix}, \quad i = 1, \ldots, p - 1,
$$

(2.19)

with $C_{i,22}$ a square matrix of order $k_{i+1}$. Let

$$
H_i = C_{i-1,22} + A_{i+1,11}, \quad i = 1, \ldots, p - 1,
$$

be nonsingular $M$-matrices and let

$$
z_i = H_i^{-1}(C_{i-1,22}y_i + A_{i+1,11}x_{i+1}), \quad i = 1, \ldots, p - 1.
$$

Then, if $z_i \leq y_i$ and $z_i \leq x_{i+1}$, the subdirect sums $C_i$ given by (2.18) are nonsingular $M$-matrices for $i = 1, \ldots, p - 1$.

Proof. It is easy to see that applying Theorem 2.3 to each consecutive pair of matrices $C_i$ we have that $C_1$, $C_2$, $\ldots$, $C_{p-1}$ are nonsingular $M$-matrices. This can be shown by induction.

We now extend Theorem 2.10 to the sub-direct sum of $p$ submatrices of a given nonsingular $M$-matrix $M$. To that end, we first define $M(S)$ a principal submatrix of $M$ with rows and columns with indices in the set of indices $S = \{i, i + 1, i + 2, \ldots, j\}$. In [2] we call these consecutive principal submatrices. For example, matrices $A$ and $B$ given by (2.12) can be expressed as submatrices of $M$ given by (2.11) as $A = M(S_1)$, $B = M(S_2)$ with $S_1 = \{1, 2\}$ and $S_2 = \{2, 3\}$. 
Theorem 2.13. Let $M$ be a nonsingular $M$-matrix. Let $A_i = M(S_i)$, $i = 1, \ldots, p$, be principal consecutive submatrices of $M$ and consider the $p - 1$ $k_i$-subdirect sums given by
\[ C_i = C_{i-1} \oplus_{k_i} A_{i+1}, \quad i = 1, \ldots, p - 1, \tag{2.20} \]
in which $C_0 = A_1$. Then each of the $k_i$-subdirect sums $C_i$ is a nonsingular $M$-matrix.

Proof. It is easy to relate the structure of each $C_i$ to that of the submatrices $A_i$ involved. We consider that $A_i$ are overlapping principal submatrices of the form (2.12) but allowing that each $A_i$ has different number of blocks. Let $M$ be partitioned as
\[ M = \begin{bmatrix}
    M_{11} & M_{12} & M_{13} & \cdots & M_{1n} \\
    M_{21} & M_{22} & M_{23} & \cdots & M_{2n} \\
    M_{31} & M_{32} & M_{33} & \cdots & M_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    M_{m1} & M_{m2} & M_{m3} & \cdots & M_{mn}
\end{bmatrix} \tag{2.21} \]
according with the size of the principal submatrices $A_i$. Each block $M_{ij}$ may be a submatrix of more than one $A_m$, $m = 1, \ldots, p$. Let $b_{ij}^{(l)} \geq 0$ be the number of matrices $A_m$ such that $M_{ij}$ is a submatrix of $A_m$ for $m = 1, \ldots, l + 1$. Of course we can have $b_{ij}^{(l)} = 0$. Let us consider the $l$th subdirect sum $C_l$, $1 \leq l \leq p - 1$, which is of the form
\[ C_l = \begin{bmatrix}
    b_{11}^{(l)} M_{11} & b_{12}^{(l)} M_{12} & b_{13}^{(l)} M_{13} & \cdots & b_{1n}^{(l)} M_{1n} \\
    b_{21}^{(l)} M_{21} & b_{22}^{(l)} M_{22} & b_{23}^{(l)} M_{23} & \cdots & b_{2n}^{(l)} M_{2n} \\
    b_{31}^{(l)} M_{31} & b_{32}^{(l)} M_{32} & b_{33}^{(l)} M_{33} & \cdots & b_{3n}^{(l)} M_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    b_{m1}^{(l)} M_{m1} & b_{m2}^{(l)} M_{m2} & b_{m3}^{(l)} M_{m3} & \cdots & b_{mn}^{(l)} M_{mn}
\end{bmatrix}. \tag{2.22} \]
Observe that $C_l$ is a $Z$-matrix and that $b_{ii}^{(l)} > 0$. Furthermore, for each column it holds that $b_{ii}^{(l)} \geq b_{jj}^{(l)}$, $j = 1, \ldots, l$.

The proof proceeds in a manner similar to that of Theorem 2.10. Consider the $Z$-matrix (partitioned in the same manner as $M$)
\[ T_l = M_l \operatorname{diag}(b_{11}^{(l)}, b_{22}^{(l)}, b_{33}^{(l)}, \ldots, b_{ll}^{(l)}), \]
where $M_l$ is the principal submatrix of (2.21) with row and column blocks from 1 to $l$. It follows that $T_l^{-1} \geq O$ and therefore $T_l$ is a nonsingular $M$-matrix. Finally, since $C_l \geq T_l$, we conclude that $C_l$ is a nonsingular $M$-matrix, $l = 1, \ldots, p$. \(\square\)

Example 2.14. Given the nonsingular $M$-matrix $M$ of Example 2.11, let us consider the following overlapping blocks
\[ A_1 = M(\{1, 2, 3\}) = \begin{bmatrix}
13/14 & -4/23 & -3/20 \\
-3/7 & 21/23 & -1/5 \\
-1/7 & -7/46 & 17/20
\end{bmatrix}. \]
Subdirect Sums of Nonsingular M-matrices and of Their Inverses

\[ A_2 = M(\{2, 3, 4, 5\}) = \begin{bmatrix} 21/23 & -1/5 & -1/21 & -1/3 \[ \begin{bmatrix} -7/46 & 17/20 & -1/14 & -1/186 \\ -27/92 & -1/15 & 4/7 & -58/93 \\ -9/46 & -3/10 & -1/7 & 53/62 \end{bmatrix} \] \]


Then we have the 2-subdirect sum

\[ C_1 = A_1 \oplus_2 A_2 = \begin{bmatrix} 13/14 & -4/23 & -3/20 & 0 & 0 \\ -3/7 & 42/23 & -2/5 & -1/21 & -1/93 \\ -1/7 & -7/23 & 17/10 & -1/14 & -1/186 \\ 0 & -27/92 & -1/15 & 4/7 & -58/93 \\ 0 & -9/46 & -3/10 & -1/7 & 53/62 \end{bmatrix} \]

which is a nonsingular M-matrix, and the 3-subdirect sum


which is also a nonsingular M-matrix in accordance with Theorem 2.13. Observe that in this example we have \( k_1 = 2 \) and \( k_2 = 3 \). Note also that, for example, we have \( b_{22}^{(1)} = 2, b_{33}^{(1)} = 2, b_{14}^{(1)} = 0, b_{22}^{(2)} = 2, b_{33}^{(2)} = 2, b_{14}^{(2)} = 3, b_{14}^{(2)} = 0 \).

3. Subdirect sums of inverses. Let \( A \) and \( B \) be nonsingular matrices partitioned as in (2.1). In this section we consider the \( k \)-subdirect sum of their inverses. We will establish counterparts to some of results in the previous sections. Let us denote by \( G = A^{-1} \oplus_k B^{-1} \), with \( A^{-1} \) and \( B^{-1} \) partitioned as in (2.3), i.e.,

\[ G = \begin{bmatrix} A_{11} & \hat{A}_{12} & 0 \\ \hat{A}_{21} & \hat{A}_{22} + \hat{B}_{11} & \hat{B}_{12} \\ 0 & \hat{B}_{21} & \hat{B}_{22} \end{bmatrix} \]

As a corollary to, and in analogy to Theorem 2.1, the next statement indicates that the nonsingularity of \( A_{22} + B_{11} \) is a necessary condition to obtain \( G \) nonsingular.

Theorem 3.1. Let \( A \) and \( B \) be nonsingular matrices partitioned as in (2.1) and let their inverses be partitioned as in (2.3). Let \( G = A^{-1} \oplus_k B^{-1} \) partitioned as in (3.1) with \( k \geq 1 \). Then \( G \) is nonsingular if and only if \( H = A_{22} + B_{11} \) is nonsingular.
We remark that in analogy to the expression (2.5) of $C^{-1}$, the explicit form of $G^{-1}$ is

$$G^{-1} = \begin{bmatrix} A_{11} - B_{11}^{-1}A_{21} & A_{12} - B_{11}^{-1}A_{22} & A_{12}^{-1}B_{12} \\ B_{11}^{-1}A_{21} & B_{11}^{-1}A_{22} & -B_{11}^{-1}B_{12} + B_{12} \\ B_{21}^{-1}A_{21} & B_{21}^{-1}A_{22} & -B_{21}^{-1}B_{12} + B_{22} \end{bmatrix}. \tag{3.2}$$

**Corollary 3.2.** When $A$ and $B$ are nonsingular $M$-matrices with the common block $A_{22} = B_{11}$ a square matrix of order $k$, i.e., of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} A_{22} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

then $H = 2A_{22}$ is nonsingular and therefore $G = A^{-1} \oplus_k B^{-1}$ is nonsingular.

We note that this is the case when $A$ and $B$ are overlapping submatrices of an $M$-matrix, i.e., of the form (2.12) and (2.11) considered in Section 2.2, where we were interested in the subdirect sum of $A$ and $B$. Here we conclude that the subdirect sum of their inverses is always nonsingular.

**Example 3.3.** Let $A$ and $B$ be the matrices of Example 2.11, then according to Corollary 3.2, the 3-subdirect sum of the inverses

$$G = A^{-1} \oplus_3 B^{-1} \approx \begin{bmatrix} 1.5033 & 0.5513 & 0.5547 & 0.2757 & 0.3912 & 0 \\ 0.9540 & 1.5996 & 0.7158 & 0.3635 & 0.4038 & 0 \\ 0.6004 & 0.5636 & 2.9750 & 0.8144 & 0.7407 & 0.3708 \\ 2.0383 & 2.1242 & 3.5729 & 6.5498 & 5.3372 & 2.0139 \\ 0.8953 & 0.9650 & 2.0470 & 1.8025 & 3.9062 & 0.9048 \\ 0 & 0 & 0.8551 & 1.3803 & 1.2652 & 1.9143 \end{bmatrix}$$

is a nonsingular matrix.

In the above example a direct computation shows that $G^{-1}$ is not an $M$-matrix:

$$G^{-1} \approx \begin{bmatrix} 0.8900 & -0.2337 & -0.0750 & -0.0119 & -0.0511 & 0.0512 \\ -0.4682 & 0.8566 & -0.1000 & -0.0238 & -0.0054 & 0.0470 \\ -0.0714 & -0.0761 & 0.4250 & -0.0357 & -0.0027 & -0.0435 \\ -0.0952 & -0.1467 & -0.0323 & 0.2357 & -0.3118 & -0.1467 \\ -0.0357 & -0.0978 & -0.1500 & -0.0714 & 0.4274 & -0.0978 \\ 0.1242 & 0.2045 & -0.0667 & -0.1429 & -0.0565 & 0.7123 \end{bmatrix}$$

which is not a $Z$-matrix. Note that when $A$ and $B$ are $M$-matrices we have from (3.1) that $G = A^{-1} \oplus B^{-1}$ is nonnegative. Therefore assuming that $G^{-1}$ exists we have $(G^{-1})^{-1} \geq 0$. Then it is a natural question to seek conditions so that $G^{-1}$ is a nonsingular $M$-matrix. We study this question next.

The expressions (3.1) of $G$ and (3.2) of $G^{-1}$, Theorem 3.1, and the observation that for nonsingular $M$-matrices we have $(G^{-1})^{-1} \geq 0$, imply the following result.

**Theorem 3.4.** Let $A$ and $B$ be nonsingular $M$-matrices partitioned as in (2.1) and their inverses partitioned as in (2.3). Let $G = A^{-1} \oplus_k B^{-1}$ with $k \geq 1$, and let $H = A_{22} + B_{11}$ be nonsingular. Then $G^{-1}$ is a nonsingular $M$-matrix if and only if $G^{-1}$ is a $Z$-matrix.
Subdirect Sums of Nonsingular M-matrices and of Their Inverses

Corollary 3.5. Let $A$ and $B$ be lower and upper block triangular nonsingular M-matrices, respectively, partitioned as in (2.10) with $A_{22}$ and $B_{11}$ square matrices of order $k$ and $H = A_{22} + B_{11}$ nonsingular. Then $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ is a nonsingular M-matrix if and only if the following conditions hold:

i) $B_{11}^{-1}A_{21} \leq O$.

ii) $B_{11}^{-1}A_{22}$ is a Z-matrix.

iii) $-B_{11}^{-1}B_{12} + B_{12} \leq O$.

Proof. >From (3.2) and (2.10) we have that

$$G^{-1} = \begin{bmatrix}
A_{11} & 0 & 0 \\
B_{11}^{-1}A_{21} & B_{11}^{-1}A_{22} & -B_{11}^{-1}B_{12} + B_{12} \\
0 & 0 & B_{22}
\end{bmatrix}$$

(3.4)

and therefore $G^{-1}$ is a Z-matrix if and only if the conditions i), ii) and iii) hold. [5]

Conditions i), ii) and iii) in the corollary are not as stringent as they may appear. For example, let $A$ and $B$ be block triangular nonsingular M-matrices partitioned as in (2.10) with a common block $A_{22} = B_{11}$, a square matrix of order $k$, i.e.,

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_{22} & B_{12} \\ 0 & B_{22} \end{bmatrix}.$$  

(3.5)

Then $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ is a nonsingular M-matrix, since we have from (3.4) that

$$G^{-1} = \begin{bmatrix}
\frac{1}{2} A_{11} & O & O \\
\frac{1}{2} A_{21} & \frac{1}{2} A_{22} & \frac{1}{2} B_{12} \\
O & O & B_{22}
\end{bmatrix},$$

and therefore $G^{-1}$ is a Z-matrix. In fact, in this case, we have

$$G = \begin{bmatrix}
-A_{11}^{-1}A_{21} & O & O \\
-A_{22}^{-1}A_{21}A_{11}^{-1} & 2A_{22}^{-1} & -A_{22}^{-1}B_{12}B_{22}^{-1} \\
O & O & B_{22}
\end{bmatrix} \geq O.$$  

The next example illustrates this situation.

Example 3.6. Let $A$ and $B$ be the matrices of Example 2.8, then

$$G = A^{-1} \oplus_2 B^{-1} = \begin{bmatrix}
1/3 & 0 & 0 & 0 \\
1/8 & 49/80 & 21/80 & 9/20 \\
7/24 & 77/80 & 73/80 & 11/10 \\
0 & 0 & 0 & 1/2
\end{bmatrix},$$

and

$$G^{-1} = \begin{bmatrix}
3 & 0 & 0 & 0 \\
-18/49 & 146/49 & -6/7 & -39/49 \\
-4/7 & -22/7 & 2 & -11/7 \\
0 & 0 & 0 & 2
\end{bmatrix}.$$
is a nonsingular $M$-matrix in accordance with Corollary 3.5.

Note that if the hypotheses of Corollary 3.5 are satisfied, and recalling Theorem 2.6, we have that each of the matrices $C = A \oplus_k B$ and $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ are both nonsingular $M$-matrices.

4. $P$-matrices. A square matrix is a $P$-matrix if all its principal minors are positive. As a consequence we have that all the diagonal entries of a $P$-matrix are positive. It is also follows that a nonsingular $M$-matrix is a $P$-matrix. It can also be shown that if $A$ is a nonsingular $M$-matrix, then $A^{-1}$ is a $P$-matrix; see, e.g., [5].

In [3] it is shown that the $k$-subdirect sum (with $k > 1$) of two $P$-matrices is not necessarily a $P$-matrix. Our results in Sections 2.1 and 3 hold for nonsingular $M$-matrices and inverses of $M$-matrices, respectively. As these two classes of matrices are subsets of $P$-matrices, it is natural to ask if similar sufficient conditions can be found so that the $k$-subdirect sum of $P$-matrices is a $P$-matrix. The following example indicates that the answer may not be easy to obtain, since even in the simplest case of diagonal submatrices the $k$-subdirect sum may not be a $P$-matrix.

**Example 4.1.** Given the $P$-matrices

$$A = \begin{bmatrix} 543 & 388 & 322 \\ 69 & 160 & 0 \\ 368 & 0 & 375 \end{bmatrix}, \quad B = \begin{bmatrix} 136 & 0 & 219 \\ 0 & 225 & 159 \\ 61 & 177 & 230 \end{bmatrix}$$

we have that the 2-subdirect sum

$$C = A \oplus_2 B = \begin{bmatrix} 543 & 388 & 322 & 0 \\ 69 & 296 & 0 & 219 \\ 368 & 0 & 600 & 159 \\ 0 & 61 & 177 & 230 \end{bmatrix}$$

is not a $P$-matrix, since $\det(C) < 0$.

**Acknowledgment.** We thank the referee for a very careful reading of the manuscript and for his comments.

REFERENCES


