AN ELEMENTARY PROOF OF THE
HILBERT–MUMFORD CRITERION*
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Abstract. An elementary proof of the Hilbert–Mumford semistability criterion is given that is valid over \( \mathbb{C} \). The proof of the criterion is deduced from an elementary lemma in linear algebra that may be of independent interest.

Key words. Semistability criterion, Algebraic one-parameter groups.

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1. Introduction and lemma. A classical result of geometric invariant theory is the Hilbert–Mumford semistability criterion. In one form, it deals with a linear action of a reductive algebraic group \( G \) on a vector space over any field \( k \). The references [1], [2], [3], [4] contain proofs over algebraically closed fields, and [5] contains a proof that works over algebraic number fields as well. Here, a transparent elementary proof is given that is valid over \( \mathbb{C} \). An elementary positivity lemma in linear algebra is proved and used to deduce the proof of the criterion over \( \mathbb{C} \). The lemma may be of independent interest.

We start with the following positivity lemma.

Lemma 1.1. Let \( m_{ij}, 1 \leq i \leq r, 1 \leq j \leq n, \) be integers satisfying the following property:

If \( b_1, \ldots, b_r \) are real numbers (not all zero) such that

\[
    b_1 m_{ij} + \cdots + b_r m_{ij} = 0 \quad \forall \ j = 1, \ldots, n,
\]

then at least two of the \( b_i \) must have opposite signs.

Then there are real numbers (and, therefore, also integers) \( c_i \) such that

\[
    m_{ii} c_1 + \cdots + m_{in} c_n > 0 \quad \forall \ i \leq r.
\]

Proof. The property above means that the kernel of the linear map \( M \) from \( \mathbb{R}^r \) to \( \mathbb{R}^n \) given by

\[
    (b_1, \ldots, b_r) \mapsto \left( \sum_{i=1}^r b_i m_{i1}, \ldots, \sum_{i=1}^r b_i m_{in} \right)
\]

intersects the “positive orthant” \( \mathcal{O} \) in \( \mathbb{R}^r \) only in zero. The assertion of the lemma amounts to the statement that the image of the transpose \( ^tM \) of \( M \) intersects the

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interior of $\mathcal{O}$. Since $\text{Ker}(M)$ and the image of $^tM$ are orthogonal complements of each other, it suffices to show that $\text{Ker}(M)^\perp$ intersects the interior of $\mathcal{O}$. We show, more generally, that if $K$ is a subspace of $\mathbb{R}^r$ intersecting $\mathcal{O}$ only in zero, then $K^\perp$ intersects the interior of $\mathcal{O}$. We will first show that $K$ can be assumed to be of codimension 1. Suppose that $K$ has codimension $k \geq 2$. Now, $D$ denotes the image of $\mathcal{O}$ in $\mathbb{R}^r/K \cong \mathbb{R}^k$. Since $\mathbb{R}^k \setminus \{0\}$ is connected, there is a vector $v \neq 0$ in $\mathbb{R}^k \setminus (D \cup -D)$. Pulling back to $\mathbb{R}^r$, we get a subspace $L$ containing $K$ in $\mathbb{R}^r$ of one more dimension such that $L \cap \mathcal{O} = \{0\}$. In the above argument, one could replace $\mathcal{O}$ more generally by a closed cone $C$ in $\mathbb{R}^r$ such that $C \cap -C = \{0\}$. We have used the fact that $D$ is again closed. Proceeding in this way, we can assume that $K$ has codimension 1. Now, let the equation of $K$ be $\sum_{i=1}^r \lambda_i X_i = 0$. Then $K \cap \mathcal{O} = \{0\}$, which evidently forces either all of the $\lambda_i > 0$ or all of the $\lambda_i < 0$. Suppose $\lambda_i > 0 \forall i$. Then $K^\perp$ is generated by the vector $(\lambda_1, \ldots, \lambda_r)$ and, obviously, $(\lambda_1, \ldots, \lambda_r)$ is in the interior of $\mathcal{O}$. This completes the proof.  

**Remark 1.2.** Note that in the above, $\mathcal{O}$ can be replaced by any closed cone subtending an angle $\geq 90^\circ$. The statement is false for cones of smaller angle.

**2. The proof of the semistability criterion.** Let us see how the lemma applies to the following statement, known as the semistability criterion.

**Theorem 2.1.** Let $G = GL(n, \mathbb{C})$ act linearly on a vector space $V$. Let $v \in V$ be a point that is not semistable, i.e., the closure $\overline{G \cdot v}$ of the orbit $G \cdot v$ (in the classical topology) contains 0. Then there exists an algebraic one-parameter subgroup $A \cong GL_1$ of $G$ such that $0 \in \overline{A \cdot v}$.

**Proof.** We have the (Cartan) decomposition $G = KTK$, where $K$ is $U(n)$ and $T$ is the maximal diagonal torus—this can be easily deduced from the spectral theorem for Hermitian operators. From this decomposition, it immediately follows that $0 \in \overline{T \cdot kv}$ for some $k \in K$. It is enough to get a multiplicative one-parameter subgroup $A$, as in the theorem, for the vector $kv$, since the group $k^{-1}Ak$ works for $v$ then. So, we rename $kv$ as $v$ and work with it without any loss of generality. Write $v = \sum_{i=1}^r v_{\chi_i}$, where

$$v_{\chi_i} \in V_{\chi_i} := \{w \in V : t \cdot w = \chi_i(t)w \ \forall \ t \in T\}$$

for some algebraic characters $\chi_i : T \to \mathbb{C}^\times$. Let $\chi_i = \sum_{j=1}^n m_{ij} \lambda_j$, where $\lambda_j : T \to \mathbb{C}^\times$ is the character $\text{diag}(t_1, \ldots, t_n) \mapsto t_j$; here $m_{ij}$ are integers. So, we have

$$tv = \sum_{i=1}^r t^{m_{i1}} \cdots t^{m_{in}} v_{\chi_i}$$

for any $t = \text{diag}(t_1, \ldots, t_n) \in T$.

**Claim.** If $b_1, \ldots, b_n$ are real numbers (not all zero) such that

$$b_1 m_{i1} + \cdots + b_r m_{ir} = 0 \ \forall \ j = 1, \ldots, n,$$

then at least two of the $b_i$'s are of opposite signs.
To prove the claim, we suppose, on the contrary, that there are \( b_i \) (not all zero) all of the same sign such that

\[
b_1 m_{1j} + \cdots + b_r m_{rj} = 0 \quad \forall j = 1, \ldots, n.
\]

Let \( t^{(k)} = \text{diag}(t_1^{(k)}, \ldots, t_n^{(k)}) \in T \) be a sequence such that \( t^{(k)}.v \to 0 \) as \( k \to \infty \).

Therefore, \( \forall i \leq r \),

\[
(t_1^{(k)})^{m_{1i}} \cdots (t_n^{(k)})^{m_{ni}} \to 0 \quad \text{as} \quad k \to \infty.
\]

Suppose, now, that \( b_1 \neq 0 \). Then

\[
-m_{1j} = \frac{b_2}{b_1} m_{2j} + \cdots + \frac{b_r}{b_1} m_{rj} \forall j \leq n
\]

so that

\[
t_1^{-m_{11}} \cdots t_n^{-m_{ni}} = (t_1^{m_{11}} \cdots t_n^{m_{ni}})^{\frac{b_1}{b_2}} \cdots (t_1^{m_{11}} \cdots t_n^{m_{ni}})^{\frac{b_r}{b_1}}.
\]

Since \( \frac{b_i}{b_j} \geq 0 \forall i \geq 2 \), and not all of them are zero, the right-hand side of (2) tends to 0 as \( (t_1, \ldots, t_n) \) runs over the sequence \( (t_1^{(k)}, \ldots, t_n^{(k)}) \). Looking at the left-hand side of (2), we have a contradiction of (1). This proves the claim.

Let us continue with the proof of the theorem. First, an application of the lemma ensures the existence of integers \( c_i \) such that

\[
m_{1i} c_1 + \cdots + m_{ni} c_n > 0 \quad \forall i \leq r.
\]

Consider the algebraic one-parameter subgroup \( GL_1 \) in \( T \) given by the homomorphism

\[
\theta : GL_1 \to T, \quad t \mapsto \text{diag}(t, \ldots, t).
\]

Note that \( \theta(t).v = \sum_{i=1}^r m_{1i} c_1 + \cdots + m_{ni} c_n \chi_i \). By (3), it is clear that \( 0 \in \theta(GL_1).v \).

The following is a corollary of the proof.

**Corollary 2.2.** Let \( (,) \) denote the nondegenerate pairing

\[
X_\ast(T) \times X(T) \to \mathbb{Z},
\]

where \( X_\ast(T) = \text{Hom}(GL_1, T) \) is the group of multiplicative one-parameter subgroups and \( X(T) = \text{Hom}(T, GL_1) \) is the character group of \( T \). Then \( \theta \in X_\ast(T) \) satisfies

\[
0 \in \theta(GL_1).v \quad \text{if and only if} \quad \theta(\chi_i) > 0 \quad \forall i \leq r.
\]

**Remark 2.3.** For the other classical groups over \( \mathbb{C} \), the proof is completely similar.

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REFERENCES