The notion of a tensor product of topological groups and modules is important in theory of topological groups, algebraic number theory. The tensor product of compact zero-dimensional modules over a pseudocompact algebra was introduced in [B] and for the commutative case in [GD], [L]. The notion of a tensor product of abelian groups was introduced in [H]. The tensor product of modules over commutative topological rings was given in [AU]. We will construct in this note the tensor product of a right compact $R$-module $AR$ and a left compact $R$-module $RB$ over a topological ring $R$ with identity. Some properties of tensor products are given.

Notation $\omega$ stands for the set of all natural numbers. If $A$ is a locally compact group, $K$ a compact subset of it and $\varepsilon > 0$, then $T(K, \varepsilon) := \{\alpha \in A^*: \alpha(K) \subseteq \varphi (O\varepsilon)\}$, where $\varphi$ is the canonical homomorphism of $R$ on $R/\mathbb{Z} = T$. If $m, n$ are natural numbers then $[m, n]$ stands for the set of all natural numbers $x$ such that $m \leq x \leq n$.

Let $R$ be a topological ring with identity and $AR$, $RB$ compact unitary right and left $R$-modules, respectively. A continuous function $\beta : A \times B \to C$, where $C$ is a compact abelian group, is said to be $R$-balanced if it is linear on each variable, i.e., $\beta(a_1 + a_2, b) = \beta(a_1, b) + \beta(a_2, b)$ and $\beta(a, b_1 + b_2) = \beta(a, b_1) + \beta(a, b_2)$ for each $a, a_1, a_2 \in A, b, b_1, b_2 \in B$, and $\beta(ar, b) = \beta(a, rb)$ for each $a \in A, b \in B, r \in R$.

A pair $(C, \pi)$ where $C$ is a compact abelian group and $\pi : A \times B \to C$ is a $R$-balanced map is called a tensor product of $AR$ and $RB$ provided for each compact abelian group $D$ and each $R$-balanced mapping $\alpha : A \times B \to D$ there exists a unique continuous homomorphism $\alpha : C \to D$ such that the following diagram commutes,

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\pi} & C \\
\alpha \downarrow & & \alpha \hat{\circ} \\
D & & \end{array}
$$

i.e., $\alpha = \alpha \hat{\circ} \pi$.

Remark. If $AR$, $RB$ are compact right and left $R$-modules, respectively, $C$ a compact abelian group, $\pi : A \times B \to C$ a $R$-balanced mapping and $\alpha : C \to C_1$ a continuous homomorphism, then $\alpha \hat{\circ} \pi : A \times B \to C_1$ is a $R$-balanced mapping.

Proposition. The tensor product, if it exists, is unique up to a topological isomorphism.

Proof. Let $(C, \pi)$ be a tensor product of $AR$ and $RB$. Then the subgroup $C_1$ of $C$ generated by elements $\pi(a, b), a \in A, b \in B$ is dense in $C$. Denote by $p$ the canonical
homomorphism of $C$ on $C/\hat{C}_1$. Consider the trivial $R$-balanced mapping $\pi_1$ of $A\times B$ in $C/\hat{C}_1$, i.e., $\pi_1(a,b) = 0$ for every $a \in A, b \in B$. Then $\pi_1 = 0 \circ \pi = p \circ \pi$. By the definition of a tensor product $p = 0$, hence $C = \hat{C}_1$.

Let now $(C, \pi)$ and $(C_1, \pi_1)$ be two tensor products of $A_R$ and $\hat{B}_R$. Then there exist continuous homomorphisms $\alpha : C \to C_1$ and $\beta : C_1 \to C$ such that $\pi_1 = \alpha \circ \pi = \beta \circ \pi$. Then $(\alpha \circ \beta)(\pi_1(a, b)) = (\alpha \circ \beta)(\pi(a, b)) = \pi_1(a, b) = \pi_1(a, b) = \pi(a, b)$ for each $a \in A, b \in B$. Hence $\alpha \circ \beta = 1_C$. In an analogous way, $(\beta \circ \alpha)(\pi(a, b)) = \beta \circ \alpha(\pi(a, b)) = \beta(\pi_1(a, b)) = \pi(a, b)$ for each $a \in A, b \in B$, hence $\beta \circ \alpha = 1_C$. Therefore, $C$ and $C_1$ are topologically isomorphic.

We will prove the existence of the tensor product for any compact right $R$-module $A_R$ and any compact left $R$-module $\hat{B}_R$. It will be denoted by $A \otimes \hat{B}_R$.

**Theorem 1.** If $A_R$ is a compact unitary right $R$-module and $\hat{B}_R$ is a compact left unitary $R$-module over a topological ring $R$ with identity then there exists the tensor product $A \otimes \hat{B}_R$.

**Proof.** Let $F$ be the discrete group of all $R$-balanced mappings $f$ of $A \times B$ in $\mathbb{T}$ having the following properties:

i) $f(ar, b) = f(a, rb)$ for all $r \in R, a \in A, b \in B$

ii) there exists a neighborhood $V$ of zero of $R$ such that $f(av, b) = 0$ for all $v \in V, a \in A, b \in B$.

Consider the dual group $C = F^*$. Define $\pi : A \times B \to C$ as follows: If $(a, b) \in A \times B$, then put $\pi(a, b)(f) := f(a, b)$ for each $f \in F$. It is easy to prove that $\pi$ is a $R$-balanced mapping. Let, for example, $a \in A, r \in R, b \in B$. Then for each $f \in F$, $\pi(ar, b)(f) = f(ar, b) = f(a, rb) = \pi(a, rb)(f)$, hence $\pi(ar, b) = \pi(a, rb).

We affirm that $\pi$ is continuous. Let $W$ be any neighborhood of zero of $C$. Then there is an $\varepsilon > 0$ and a finite subset $K$ of $F$ such that $T(K, \varepsilon) \subseteq W$. Since all $f \in K$ are continuous at $(0, 0)$, there exist neighborhoods $U, V$ of zeros of $A$ and $B$, respectively, such that $f(U \times V) \subseteq \varphi(O_1)$ for all $f \in K$. Then $\pi(U \times V) \subseteq W$. Indeed, if $f \in K, u \in U, v \in V$, then $\pi(u, v)(f) = f(u, v) \subseteq \varphi(O_1)$, and so $\pi(u, v) \subseteq T(K, \varepsilon)$. We proved that $\pi(U \times V) \subseteq W$, hence $\pi$ is continuous at $(0,0)$.

Let $a \in A, K$ a finite subset of $F$ and $\varepsilon > 0$. Since every $f \in K$ is continuous there exists a neighborhood $V$ of zero of $B$ such that $f(a, V) \subseteq \varphi(O_1)$ for each $f \in K$. Then $\pi(a, V) \subseteq T(K, \varepsilon)$. Indeed, if $v \in V$, then for each $f \in K, \pi(a, v)(f) = f(a, v) \subseteq \varphi(O_1)$. Therefore $\pi(a, V) \subseteq T(K, \varepsilon)$, i.e., $\pi$ is continuous at $(a, 0)$. By symmetry $\pi$ is continuous at $(0, b), b \in B$. We proved that $\pi$ is a continuous $R$-balanced map.

We will prove now that $C$ is the tensor product of $A$ and $B$. Let $\alpha : A \times B \to X$ be a $R$-balanced map in a compact abelian group $X$. We define a homomorphism $\lambda : X^* \to F$ as follows: for every $y \in X^*, \gamma \circ \alpha : A \times B \to T$ is a $R$-balanced mapping of $A \times B$. Theorem 1
in $T$, i.e., $\gamma^*a \in F$. Put $\lambda(\gamma) = \gamma^*a$, $\gamma \in X^*$. We claim that $\lambda$ is a homomorphism. Indeed, let $\gamma_1, \gamma_2 \in X^*$. Then for each $a \in A, b \in B$, $\lambda(\gamma_1 + \gamma_2)(a, b) = (\gamma_1 + \gamma_2)(\lambda(a, b)) = \gamma_1(\lambda(a, b)) + \gamma_2(\lambda(a, b)) = \lambda(\gamma_1)(a, b) + \lambda(\gamma_2)(a, b)$.

Let $\lambda^*: F^* \to X^{**}$ be the conjugate homomorphism for $\lambda$. Put $\hat{\lambda} : F^* \to X$, $\hat{\lambda} = \omega^{-1}\lambda^*$, where $\omega$ is the canonical topological isomorphism of $X$ on $X^{**}$. We affirm that $\alpha = \hat{\lambda} \circ \pi$. Indeed, fix $(a, b) \in A \times B$. Then $\alpha(a, b) = \hat{\lambda}((\pi(a, b))) = \omega^{-1}\lambda^*((\pi(a, b))) = \omega(\alpha(a, b)) = \lambda^*((\pi(a, b))) = \pi(a, b) \circ \lambda$. The last equality is true $\iff$ for each $\gamma \in X^*$, $\omega(\alpha(a, b))(\gamma) = \pi(a, b)(\gamma) \Rightarrow \gamma(a, b) = \gamma(a, b)(\gamma) \Rightarrow \gamma(a, b) = \gamma(\alpha(a, b)) = \gamma(a, b)(\gamma^*a) \Rightarrow \gamma(a, b) = \gamma(a, b)(\gamma^*a)$, which is true.

The uniqueness of $\hat{\lambda}$. It is sufficient to prove that the set $\{ \pi(a, b) : a \in A, b \in B \}$ generates $A \otimes_R B$ as a topological group. It is well known from the duality theory that if $X$ is a locally compact abelian group and $S$ a subgroup of $X^*$ which separates points then $S$ is dense in $X^*$. We affirm that the subgroup $D = \{ \pi(a, b) : a \in A, b \in B \}$ separates points of $F$. Indeed, let $0 \neq \xi \in C$, then there exists $(a, b) \in A \times B$ such that $0 \neq \xi(a, b) = \pi(a, b)(\xi)$, i.e., $D$ separates points of $F$. Therefore, $\alpha$ is unique.

We will denote below $\pi(a, b)$, where $a \in A, b \in B$ by $a \otimes b$.

**Theorem 2.** If $A$, $B$ are zero-dimensional compact right and left $R$-modules then $A \otimes_R B$ is zero-dimensional.

**Proof.** Let $f \in F$; then $f$ is a continuous $R$-balanced map of $A \times B$ in $T$. Let $V$ be a neighborhood of zero of $T$ which does not contain a non-zero subgroup. For every $a \in A$ there exist a neighborhood $U_a$ and an open subgroup $V^a$ of $B$ such that $f(U_a \times V^a) \subseteq V$. There exist $a_1, \ldots, a_n \in A$ such that $A = U_{a_1} \cup \ldots \cup U_{a_n}$. Denote $V_0 = V^{a_1} \cap \ldots \cap V^{a_n}$. We obtain immediately that $f(A, V_0) \subseteq V$, hence $f(A, V_0) = 0$.

Let $B = V_0 \cup (b_1 + V_0) \cup \ldots \cup (b_k + V_0)$. Then $f(A \times B) \subseteq f(A, b_1) + \ldots + f(A, b_k)$. Each subset $f(A, b_1), \ldots, f(A, b_k)$ is a compact zero-dimensional subgroup of $T$. Therefore $f(A, b_1) + \ldots + f(A, b_k)$ is a finite subgroup of $T$. It follows that there exists $m \in \omega$ such that $mf = 0$, i.e., $F$ is a torsion group. It is well known that $F^*$ is zero-dimensional.

The author learned recently that a particular analogue of Theorem 2 was proved by Hofmann (see, [HM]).

**Theorem 3.** If $A_R$ or $B_R$ is connected then $A \otimes_R B = 0$.

**Proof.** Assume that $B$ is connected. Let $V$ be a neighborhood of zero of $T$ which does not contain non-zero subgroups. Fix $\xi \in C$. There exists a neighborhood $V_0$ of $0$ of $B$
such that $\xi(A \times V_0) = 0$ (as in the proof of the previous theorem). Since $B$ is generated by $V_0$, $\xi(A \times B) = 0$. We obtained that $F = 0$, hence $C = 0$.

Let $A_R$ be a compact right $R$-module and $B_L$ a compact left $R$-module over a topological ring $R$. If $X \subseteq A$, $Y \subseteq B$, then we will denote by $[X \otimes Y]$ the closure of the subgroup of $A \otimes B_L$ generated by elements of the form $\sum_{i=0}^{n} x_i \otimes y_i$, $x_i \in X$, $y_i \in Y$, $n \in \omega$.

**Theorem 4.** If $A_R$ and $B_L$ are compact zero-dimensional left and right $R$-modules then the family $[U \otimes B_L] + [A \otimes V]$ where $U$ runs all open subgroups of $A$ and $V$ runs all open subgroups of $B$ is a fundamental system of neighborhoods of zero of $A \otimes B_L$.

**Proof.** The subgroup $\langle x \otimes y : x \in A, y \in B \rangle$ is dense in $A \otimes B_L$. Let $U$ be an open subgroup of $A$ and $V$ an open subgroup of $B$. There exist finite symmetric subsets $F \subseteq A$, $K \subseteq B$ such that $A = F + U$, $B = K + V$. For each $x \in F$, $y \in K$, $u \in U$, $v \in V$, $(x + u) \otimes (y + v) = x \otimes y + x \otimes v + u \otimes y + u \otimes v$. Since $A/U$ is finite, there exists $k \in \omega$ such that $kA \subseteq U$. Consider the finite subsets $H = \{ (lx) \otimes y : l \in [0, k-1], x \in F, y \in K \}$, $H_1 = [l]H$. It is evidently that $C = H_1 + [U \otimes B_L] + [A \otimes B_L]$, hence $[U \otimes B_L] + [A \otimes B_L]$ is open.

Let $W$ be an open subgroup of $A \otimes B_L$. By continuity of the mapping $\pi : A \times B \rightarrow A \otimes B_L$ and compactness of $A$ and $B$ there exist an open subgroup $U$ of $A$ and an open subgroup $V$ of $B$ such that $U \otimes B_L \subseteq W$, $A \otimes V \subseteq W \Rightarrow [U \otimes B] + [A \otimes V] \subseteq W$.

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**References**


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