ABOUT SOME INEQUALITIES CONCERNING
THE FRACTIONAL PART

by
Alexandru Gica

Abstract. The main purpose of this paper is to find the rational numbers x which have the
property that \( \{2^n \cdot x\} \geq \frac{1}{3}, \forall n \in \mathbb{N} \).

Key words: fractional part, length.

INTRODUCTION

If \( n \in \mathbb{N}, n \geq 2 \) has the standard decomposition \( n = p_1^{a_1} \ldots p_r^{a_r} \), we define the
length of n to be the number \( \Omega(n) = \sum_{i=1}^{n} a_i, \Omega(1) = 0 \). In [1] and [2] I showed that
\( \forall n \in \mathbb{N}, n > 3, \) there exists the positive integers \( a, b \) such that \( n = a + b \) and \( \Omega(ab) \) is
an even number. The second proof from [2] uses the following lemma: if \( r \in \mathbb{N}^*, n \in \mathbb{N} \), and \( p_j \) has the usual meaning (the \( j \)-th prime number) and \( p \) is a prime number \( p \equiv \pm 3 \),
there exist the natural numbers \( a_j, j = 1, r \) such that

\[ \{ p_1^{a_1} \ldots p_r^{a_r} \frac{a}{p} \} \leq \frac{1}{p_{r+1}}. \]

If \( r = 1 \), it results that there is an \( a \in \mathbb{N} \) such that \( \{2^a \cdot \frac{a}{p} \} \leq \frac{1}{3} \). Starting from
this point I posed the problem of finding the rational numbers \( x \) such that \( \{2^n \cdot x\} \geq \frac{1}{3}, \forall n \in \mathbb{N} \).
THE MAIN RESULTS

It is enough to consider the case when $x = \frac{n}{k}$ ($n, k \in \mathbb{N}, (n, k) = 1$) is a rational number $0 < x < 1$. After some multiplications with 2, we can suppose that $k$ and $n$ are odd numbers. If $1 > x > \frac{1}{2}$, then $\{2x\} = 2x - 1 < x$ and, after some multiplications with 2, then we can suppose that $\frac{1}{3} \leq x < \frac{1}{2}$. We will prove now the main statement of the paper.

**Proposition 1**

Let $x$ a rational number $x = \frac{n}{k}$, where $n, k$ are coprime, odd natural numbers. The number $x$ has the property that:

$$\frac{1}{3} \leq x < \frac{1}{2}$$

and

$$\{2^m x\} \geq \frac{1}{3}, \forall m \in \mathbb{N}.$$

Then

$$x = \frac{2^{a_r} + 2^{a_{r-1}} + \ldots + 2^{a_1} + 2^{a_0}}{2^{a_r+2} - 1}$$

where

$$a_0 = 0 < a_1 < a_2 < \ldots < a_r$$

are natural numbers which satisfy the inequalities

$$a_{i+1} - a_i \leq 2, \forall i = 0, r - 1.$$

$r \in \mathbb{N}$ and $a_r + 2$ is the smallest number $l \in \mathbb{N}^*$ for which

$$2^l \equiv 1.$$

**Proof.** We show by induction that $\forall m \in \mathbb{N}$ we have

$$[2^{m+2} x] = 2^{b_r} + 2^{b_{r-1}} + \ldots + 2^{b_1} + 2^{b_0},$$

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where \( b_0 < b_1 < b_2 \ldots < b_r = m \) are natural numbers depending on \( m \) and satisfying the inequalities

\[
 b_{i+1} - b_i \leq 2, \quad \forall \ i = 0, r-1.
\]

\( b_0 \) could be only 0 or 1. For \( m = 0 \) the statement is obvious since \([4x] = 1\); the last equality holds since

\[
 \frac{1}{3} \leq x < \frac{1}{2}.
\]

The same inequality shows that \([8x] = 2 \) or \([8x] = 3 = 2 + 1\). This means that the statement is true for \( m = 1 \). Let us suppose that the statement is true for \( m \in \mathbb{N}^* \) and we want to prove the statement for \( m + 1 \). Using the induction hypothesis we infer that

\[
 2^{m+2}x = \lfloor 2^{m+2}x \rfloor + \{2^{m+2}x\} = 2^{b_r} + 2^{b_{r-1}} + \ldots + 2^{b_0} + \{2^{m+2}x\},
\]

where \( b_0 < b_1 < b_2 \ldots < b_r = m \) are natural numbers depending on \( m \) and satisfying the inequalities

\[
 b_{i+1} - b_i \leq 2, \quad \forall \ i = 0, r-1.
\]

\( b_0 \) could be only 0 or 1. We analyze first the case

\[
 \{2^{m+2}x\} < \frac{1}{2}.
\]

We will show that in this case \( b_0 = 0 \). Let us suppose that \( b_0 = 1 \). It results that

\[
 2^{m+1}x = 2^{b_r-1} + 2^{b_{r-1}-1} + \ldots + 2^{b_0-1} + \frac{1}{2} \{2^{m+2}x\}.
\]

From this last equality we obtain (taking into account that \( b_0=1 \) and \( \{2^{m+2}x\} < \frac{1}{2} \)) that

\[
 \{2^{m+1}x\} < \frac{1}{2} \{2^{m+2}x\} < \frac{1}{4}.
\]

The last inequality is impossible since from the hypothesis we know that

\[
 \{2^{m+1}x\} \geq \frac{1}{2}.
\]
Therefore \( b_0 = 0 \). From the above equalities we obtain that

\[
2^{m+3}x = 2^{b_r+1} + 2^{b_{r-1}+1} + \ldots + 2^{b_1+1} + 2^{b_0+1} + 2\{2^{m+2}x\},
\]

which lead us (taking into account the fact that that \( \{2^{m+2}x\} < \frac{1}{2} \) at the conclusion that

The properties of numbers \( b_i \) together with \( b_0 = 0 \) (then \( b_0 + 1 = 1 \)) ensure us that the induction step is true in this case. We have to analyze the case

\[
\{2^{m+2}x\} \geq \frac{1}{2}.
\]

Using again the equality

\[
2^{m+3}x = 2^{b_r+1} + 2^{b_{r-1}+1} + \ldots + 2^{b_1+1} + 2^{b_0+1} + 2\{2^{m+2}x\},
\]

we obtain that

The properties of numbers \( b_i \) ensure us that also in this case the induction step is proved. We showed therefore by induction that \( \forall m \in \mathbb{N} \) we have the identity

\[
\lfloor 2^{m+2}x \rfloor = 2^{b_r} + 2^{b_{r-1}} + \ldots + 2^{b_1} + 2^{b_0},
\]

where \( b_0 < b_1 < b_2 \ldots < b_r = m \) are natural numbers depending on \( m \) and satisfying the inequalities

\[
b_{i+1} - b_i \leq 2, \quad \forall i = 0, r - 1.
\]

\( b_0 \) could be only 0 or 1. Let \( a_r + 2 \) the smallest \( l \in \mathbb{N}^* \) such that

\[
2^l = 1.
\]

\( a_r + 2 \) exists since \( k \) is odd. We have \( a_r + 2 \geq 2 \) since \( k \neq 1 \) (do not forget that \( x = \frac{a}{k} \), \( n \) and \( k \) being coprime odd natural numbers; also we have \( \frac{1}{2} \leq x < \frac{1}{2} \). Since \( 2^{a_r+2} \equiv 1 \) it follows that

\[
\{2^{a_r+2}\frac{a}{k}\} = \{\frac{a}{k}\} = \{x\} = x = 2^{a_r+2}x - \lfloor 2^{a_r+2}x \rfloor.
\]

Taking into account these equalities and the statement proved above by induction, it results that

\[
x = \frac{2^{a_r} + 2^{a_{r-1}} + \ldots + 2^{a_1} + 2^{a_0}}{2^{a_r+2} - 1}
\]

where

\[
a_0 < a_1 < a_2 < \ldots < a_r
\]

are natural numbers which satisfy the inequalities

\[
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\]
$a_{i+1} - a_i \leq 2, \forall i = 0, r - 1$.

$a_0$ is 0 or 1. We have to show that $a_0 = 0$. This result from

$$\{2^{a_{r+2}}x\} = x < \frac{1}{2}$$

and from the first case of the induction above.

We will show now that if

$$x = \frac{2^{a_r} + 2^{a_{r-1}} + ... + 2^{a_1} + 2^{a_0}}{2^{a_{r+2}} - 1}$$

where

$a_0 = 0 < a_1 < a_2 < ... < a_r$

are natural numbers which satisfy the inequalities

$a_{i+1} - a_i \leq 2, \forall i = 0, r - 1 (r \in N)$,

then

$$\{2^m x\} \geq \frac{1}{2}, \forall m \in N.$$  

For proving this statement it is enough to show that the number

$$y = \frac{2^{b_r} + 2^{b_{r-1}} + ... + 2^{b_1} + 2^{b_0}}{2^{b_{r+2}} - 1}$$

(where

$b_0 = 0 < b_1 < b_2 <... < b_s$

are natural numbers which satisfy the inequalities

$b_{i+1} - b_i \leq 2, \forall i = 0, s - 1; s \in N$)

has the property that

$$\frac{1}{3} \leq y < \frac{1}{2}.$$  

We have

$$y \leq \frac{2^{b_r} + 2^{b_{r-1}} + ... + 2^0}{2^{b_{r+2}} - 1} = \frac{2^{b_r + 1} - 1}{2^{b_{r+2}} - 1} < \frac{1}{2}.$$  

For showing the second inequality we will consider two cases. The first one is when $b_r = 2l; l \in N$. In this case we have
\[ y \geq \frac{2^{2l} + 2^{2l-2} + \ldots + 2^2 + 1}{2^{2l+2} - 1} = \frac{1}{3} \]

If \( b_j = 2l + 1; l \in \mathbb{N} \) then \( y \geq \frac{2^{2l+1} + 2^{2l-1} + \ldots + 1}{3(2^{2l+3} - 1)} \geq \frac{1}{3} \).

Using the same arguments as in the above Proposition we can show the following result:

**Proposition 2**

Let \( x \) a rational number, \( x = \frac{n}{k} \), where \( n, k \) are coprime odd natural numbers. We suppose that \( x \) has the following property:

\[ \frac{1}{5} \leq x < \frac{1}{4} \]

and

\[ \{2^m x\} \geq \frac{1}{5}, \quad \forall m \in \mathbb{N}. \]

Then

\[ x = \frac{2^{a_r} + 2^{a_{r-1}} + \ldots + 2^{a_1} + 2^{a_0}}{2^{a_{r+3}} - 1} \]

where

\[ a_0 = 0 < a_1 < a_2 < \ldots < a_r \]

are natural numbers which satisfy the inequalities

\[ a_{i+1} - a_i \leq 3, \quad \forall i = 0, r (r \in \mathbb{N}), \]

If there is an \( i \) \((0 \leq i \leq r)\) such that

\[ a_{i+1} - a_i = 3, \]

then

\[ a_{i+1} = a_i + 1. \]

We denote

\[ a_{r+1} = a_r + 3; a_{r-1} = 0. \]

It will results that
The number $a_r + 3$ is the order of 2 in $U(\mathbb{Z}_k)$.  

**Proof:** The proof is similar with that of Proposition 1. The fact that $a_{r+1} = a_r + 3$ follows from the inequalities  
\[
\frac{1}{5} \leq x < \frac{1}{5}.
\]

The only fact which has to be proved is that for any $i$ ($0 \leq i \leq r$) such that  
\[
a_{i+1} - a_i = 3,
\]
then  
\[
a_{i+1} = a_i + 1.
\]

Replacing $x$ by $\{ 2^{a_{i+1}} x \}$, we observe that it is enough to show the statement only for $i = r$. We have to show that $a_{r+1} = a_r - 1$. Let us suppose that  
\[
a_{r+1} \leq a_r - 2.
\]

Then  
\[
x \leq \frac{2^{a_r} + 2^{a_r-1} - 1}{2^{a_r+3} - 1} < \frac{1}{5}.
\]

The last inequality is equivalent with  
\[
2^{a_r+3} \cdot 5 \cdot 2^{a_r} \cdot 2 \cdot 5 \cdot 2^{a_r-1} + 4 > 0
\]
and  
\[
2^{a_r-1} + 4 > 0.
\]

The last inequality is obviously true since $a_r \geq 1$ (if $a_r = 0$ then $x = \frac{1}{5} < \frac{1}{5}$; this is impossible). We obtained a contradiction since $x$ is greater than $\frac{1}{5}$. The second part of the proof is identical with that of Proposition 1.

**References**


Author:
Alexandru Gica
Faculty of Mathematics
University of Bucharest
Str. Academiei 14
Bucharest 1, Romania 010014
alex@al.math.unibuc.ro