PROBABILITY
ON TREES
AND NETWORKS

Russell Lyons
with Yuval Peres
A love and respect of trees has been characteristic of mankind since the beginning of human evolution. Instinctively, we understood the importance of trees to our lives before we were able to ascribe reasons for our dependence on them.

# Table of Contents

**Preface** vii

**Chapter 1: Some Highlights** 1

1. Branching Number 2
2. Electric Current 5
3. Random Walks 6
4. Percolation 8
5. Branching Processes 9
6. Random Spanning Trees 10
7. Hausdorff Dimension 13
8. Capacity 14
9. Embedding Trees into Euclidean Space 16

**Chapter 2: Random Walks and Electric Networks** 18

1. Circuit Basics and Harmonic Functions 18
2. More Probabilistic Interpretations 23
3. Network Reduction 26
4. Energy 30
5. Transience and Recurrence 36
6. Flows, Cutsets, and Random Paths 45
7. Trees 50
8. Growth of Trees 53
9. Cayley Graphs 58
10. Notes 61
11. Additional Exercises 63

**Chapter 3: Infinite Electrical Networks and Dirichlet Functions** 72

1. Free and Wired Electrical Currents 72
2. Planar Duality 74
3. Harmonic Dirichlet Functions 78
4. Planar Graphs and Hyperbolic Lattices 85
5. Notes 91
6. Additional Exercises 92
Chapter 4: Percolation on Trees

1. Galton-Watson Branching Processes
2. The First-Moment Method
3. The Weighted Second-Moment Method
4. Reversing the Second-Moment Inequality
5. Galton-Watson Networks
6. Additional Exercises

Chapter 5: Isoperimetric Inequalities

1. Flows and Submodularity
2. Spectral Radius
3. Planar Graphs
4. Profiles and Transience
5. Notes
6. Additional Exercises

Chapter 6: Percolation on Transitive Graphs

1. Groups and Amenability
2. Tolerance, Ergodicity, and Harris’s Inequality
3. The Number of Infinite Clusters
4. Inequalities for $p_c$
5. Merging Infinite Clusters and Invasion Percolation
6. Upper Bounds for $p_u$
7. Lower Bounds for $p_u$
8. Notes
9. Additional Exercises

Chapter 7: The Mass-Transport Technique and Percolation

1. The Mass-Transport Principle for Cayley Graphs
2. Beyond Cayley Graphs: Unimodularity
3. Infinite Clusters in Invariant Percolation
4. Critical Percolation on Nonamenable Transitive Unimodular Graphs
5. Percolation on Planar Cayley Graphs
6. Properties of Infinite Clusters
7. Notes
8. Additional Exercises

Chapter 8: Uniform Spanning Trees

1. Generating Uniform Spanning Trees
2. Electrical Interpretations
3. The Square Lattice $\mathbb{Z}^2$
4. Notes
5. Additional Exercises
Chapter 9: Uniform Spanning Forests 224

1. Basic Properties 224
2. Tail Triviality 237
3. The Number of Trees 239
4. The Size of the Trees 249
5. Loop-Erased Random Walk and Harmonic Measure From Infinity 261
6. Open Questions 263
7. Notes 263
8. Additional Exercises 264

Chapter 10: Minimal Spanning Forests 267

1. Minimal Spanning Trees 267
2. Deterministic Results 268
3. Basic Probabilistic Results 272
4. Tree Sizes 273
5. Planar Graphs 279
6. Notes 280
7. Additional Exercises 281

Chapter 11: Limit Theorems for Galton-Watson Processes 282

1. Size-biased Trees and Immigration 282
2. Supercritical Processes: Proof of the Kesten-Stigum Theorem 286
3. Subcritical Processes 289
4. Critical Processes 291
5. Decomposition 293
6. Notes 296
7. Additional Exercises 296

Chapter 12: Speed of Random Walks 298

1. The Varopoulos-Carne Bounds 298
2. Branching Number of a Graph 300
3. Stationary Measures on Trees 302
4. Galton-Watson Trees 309
5. Biased Random Walks on Galton-Watson Trees 316
6. Notes 316
7. Additional Exercises 316

Chapter 13: Hausdorff Dimension 318

1. Basics 318
2. Coding by Trees 322
3. Galton-Watson Fractals 328
4. Hölder Exponent 331
5. Derived Trees 333
6. Additional Exercises 336
## Chapter 14: Capacity 338

1. Definitions 338
2. Percolation on Trees 341
3. Euclidean Space 342
4. Generalized Diameters and Average Meeting Height on Trees 345
5. Notes 348
6. Additional Exercises 349

## Chapter 15: Harmonic Measure on Galton-Watson Trees 351

1. Introduction 351
2. Markov Chains on the Space of Trees 352
3. The Hölder Exponent of Limit Uniform Measure 357
4. Dimension Drop for Other Flow Rules 360
5. Harmonic-Stationary Measure 361
6. Calculations 366
7. Additional Exercises 371

## Comments on Exercises 374

## Bibliography 402

## Glossary of Notation 419
Preface

This book began as lecture notes for an advanced graduate course called “Probability on Trees” that Lyons gave in Spring 1993. We are grateful to Rabi Bhattacharya for having suggested that he teach such a course. We have attempted to preserve the informal flavor of lectures. Many exercises are also included, so that real courses can be given based on the book. Indeed, previous versions have already been used for courses or seminars in several countries. The most current version of the book can be found on the web. A few of the authors’ results and proofs appear here for the first time. At this point, almost all of the actual writing was done by Lyons. We hope to have a more balanced co-authorship eventually.

This book is concerned with certain aspects of discrete probability on infinite graphs that are currently in vigorous development. We feel that there are three main classes of graphs on which discrete probability is most interesting, namely, trees, Cayley graphs of groups (or more generally, transitive, or even quasi-transitive, graphs), and planar graphs. Thus, this book develops the general theory of certain probabilistic processes and then specializes to these particular classes. In addition, there are several reasons for a special study of trees. Since in most cases, analysis is easier on trees, analysis can be carried further. Then one can often either apply to other situations the results from trees or can transfer to other situations the techniques developed by working on trees. Trees also occur naturally in many situations, either combinatorially or as descriptions of compact sets in euclidean space $\mathbb{R}^d$. (More classical probability, of course, has tended to focus on the special and important case of the euclidean lattices $\mathbb{Z}^d$.)

It is well known that there are many connections among probability, trees, and groups. We survey only some of them, concentrating on recent developments of the past fifteen years. We discuss some of those results that are most interesting, most beautiful, or easiest to prove without much background. Of course, we are biased by our own research interests and knowledge. We include the best proofs available of recent as well as classic results. Much more is known about almost every subject we present. The only prerequisite is knowledge of conditional expectation with respect to a $\sigma$-algebra, and even that is rarely used. For part of Chapter 12 and all of Chapter 15, basic knowledge of ergodic theory is also required.

Exercises that appear in the text, as opposed to those at the end of the chapters, are ones that ought to be done when they are reached, as they are either essential to a proper understanding or will be used later in the text.

Some notation we use is $\langle \cdots \rangle$ for a sequence (or, sometimes, more general function), $\upharpoonright$ for the restriction of a function or measure to a set, $\mathbb{E}[X \mid A]$ for the expectation of $X$ on the event $A$, and $|\cdot|$ for the cardinality of a set. Some definitions are repeated in different
chapters to enable more selective reading.

A question labelled as **Question m,n** is one to which the answer is unknown, where \( m \) and \( n \) are numbers. Unattributed results are usually not due to us.

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**Russell Lyons**, Department of Mathematics, Indiana University, Bloomington, IN 47405-5701, USA  
rdlyons@indiana.edu  
http://mypage.iu.edu/~rdlyons/

**Yuval Peres**, Departments of Statistics and Mathematics, University of Calif., Berkeley, CA 94720-3860, USA  
peres@stat.berkeley.edu  
http://www.stat.berkeley.edu/~peres/
Chapter 1

Some Highlights

This chapter gives some of the highlights to be encountered in this book. Some of the topics in the book do not appear at all here since they are not as suitable to a quick overview. Also, we concentrate in this overview on trees since it is easiest to use them to illustrate most of the themes.

Notation and terminology for graphs is the following. A graph is a pair $G = (V, E)$, where $V$ is a set of vertices and $E$ is a symmetric subset of $V \times V$, called the edge set. The word “symmetric” means that $(x, y) \in E$ iff $(y, x) \in E$; here, $x$ and $y$ are called the endpoints of $(x, y)$. The symmetry assumption is usually phrased by saying that the graph is undirected or that its edges are unoriented. Without this assumption, the graph is called directed. If we need to distinguish the two, we write an unoriented edge as $[x, y]$, while an oriented edge is written as $(x, y)$. An unoriented edge can be thought of as the pair of oriented edges with the same endpoints. If $(x, y) \in E$, then we call $x$ and $y$ adjacent or neighbors, and we write $x \sim y$. The degree of a vertex is the number of its neighbors. If this is finite for each vertex, we call the graph locally finite. If we have more than one graph under consideration, we distinguish the vertex and edge sets by writing $V(G)$ and $E(G)$. A path in a graph is a sequence of vertices where each successive pair of vertices is an edge in the graph. A graph is connected if there is a path from any of its vertices to any other.

If there are numbers (weights) $c(e)$ assigned to the edges $e$ of a graph, the resulting object is called a network. Sometimes we work with more general objects than graphs, called multigraphs. A multigraph is a pair of sets, $V$ and $E$, together with a pair of maps from $E \to V$ denoted $e \mapsto e^-$ and $e \mapsto e^+$. The images of $e$ are called the endpoints of $e$, the former being its tail and the latter its head. If $e^- = e^+ = x$, then $e$ is a loop at $x$. Edges with the same pair of endpoints are called parallel or multiple. Given a network $G = (V, E)$ with weights $c(\cdot)$ and a subset of its vertices $K$, the induced subnetwork $G|K$ is the subnetwork with vertex set $K$, edge set $(K \times K) \cap E$, and weights $c|(K \times K) \cap E$.

Items such as theorems are numbered in this book as $C.n$, where $C$ is the chapter number and $n$ is the item number in that chapter; $C$ is omitted when the item appears
in the same chapter as the reference to it. In particular, in this chapter, items to be encountered later are numbered as they appear later.

§1.1. Branching Number.

A tree is called locally finite if the degree of each vertex is finite (but not necessarily uniformly bounded). Our trees will be rooted, meaning that some vertex is designated as the root, denoted $o$. We imagine the tree as growing (upwards) away from its root. Each vertex then has branches leading to its children, which are its neighbors that are further from the root. For the purposes of this chapter, we do not allow the possibility of leaves, i.e., vertices without children.

How do we assign an average branching number to an arbitrary infinite locally finite tree? If the tree is a binary tree, as in Figure 1.1, then clearly the answer will be “2”. But in the general case, since the tree is infinite, no straight average is available. We must take some kind of limit or use some other procedure.

\begin{figure}
\centering
\includegraphics[width=0.3\textwidth]{binary_tree.png}
\caption{The binary tree.}
\end{figure}

One simple idea is as follows. Let $T_n$ be the set of vertices at distance $n$ from the root, $o$. Define the lower growth rate of the tree to be

$$\underline{\text{gr}} T := \liminf_{n \to \infty} |T_n|^{1/n}.$$ 

This certainly will give the number “2” to the binary tree. One can also define the upper growth rate

$$\overline{\text{gr}} T := \limsup_{n \to \infty} |T_n|^{1/n}$$

and the growth rate

$$\text{gr} T := \lim_{n \to \infty} |T_n|^{1/n}$$

when the limit exists. However, notice that these notions of growth barely account for the structure of the tree: only $|T_n|$ matters, not how the vertices at different levels are connected to each other. Of course, if $T$ is spherically symmetric, meaning that for
§1. Branching Number

Each $n$, every vertex at distance $n$ from the root has the same number of children (which may depend on $n$), then there is really no more information in the tree than that contained in the sequence $\langle |T_n| \rangle$. For more general trees, however, we will use a different approach.

Consider the tree as a network of pipes and imagine water entering the network at the root. However much water enters a pipe leaves at the other end and splits up among the outgoing pipes (edges). Consider the following sort of restriction: Given $\lambda \geq 1$, suppose that the amount of water that can flow through an edge at distance $n$ from $o$ is only $\lambda^{-n}$. If $\lambda$ is too big, then perhaps no water can flow. In fact, consider the binary tree. A moment’s thought shows that water can still flow throughout the tree provided that $\lambda \leq 2$, but that as soon as $\lambda > 2$, then no water at all can flow. Obviously, this critical value of 2 for $\lambda$ is the same as the branching number of the binary tree. So let us make a general definition: the branching number of a tree $T$ is the supremum of those $\lambda$ that admit a positive amount of water to flow through $T$; denote this critical value of $\lambda$ by $\text{br} T$. As we will see, this definition is the exponential of what Furstenberg (1970) called the “dimension” of a tree, which is the Hausdorff dimension of its boundary.

It is not hard to check that $\text{br} T$ is related to $\text{gr} T$ by

$$\text{br} T \leq \text{gr} T. \quad (1.1)$$

Often, as in the case of the binary tree, equality holds here. However, there are many examples of strict inequality.

▶ Exercise 1.1.
Prove (1.1).

▶ Exercise 1.2.
Show that $\text{br} T = \text{gr} T$ when $T$ is spherically symmetric.

Example 1.1. If $T$ is a tree such that vertices at even distances from $o$ have 2 children while the rest have 3 children, then $\text{br} T = \text{gr} T = \sqrt{6}$. It is easy to see that $\text{gr} T = \sqrt{6}$, whence by (1.1), it remains to show that $\text{br} T \geq \sqrt{6}$, i.e., given $\lambda < \sqrt{6}$, to show that a positive amount of water can flow to infinity with the constraints given. Indeed, we can use the water flow with amount $6^{-n/2}$ flowing on those edges at distance $n$ from the root when $n$ is even and with amount $6^{-(n-1)/2}/3$ flowing on those edges at distance $n$ from the root when $n$ is odd.

Example 1.2. Let $T$ be a tree embedded in the upper half plane with $o$ at the origin. List $T_n$ in clockwise order as $\langle x_1^n, \ldots, x_2^n \rangle$. Let $x_k^n$ have 1 child if $k \leq 2^{n-1}$ and 3 children
otherwise; see Figure 1.2. Now, a ray is an infinite path from the root that doesn’t backtrack. If $x$ is a vertex of $T$ that does not have the form $x_{2n}^n$, then there are only finitely many rays that pass through $x$. This means that water cannot flow to infinity through such a vertex $x$ when $\lambda > 1$. That leaves only the possibility of water flowing along the single ray consisting of the vertices $x_{2n}^n$, which is impossible too. Hence $\text{br } T = 1$, yet $\text{gr } T = 2$.

![Figure 1.2. A tree with branching number 1 and growth rate 2.](image)

**Example 1.3.** If $T^{(1)}$ and $T^{(2)}$ are trees, form a new tree $T^{(1)} \vee T^{(2)}$ from disjoint copies of $T^{(1)}$ and $T^{(2)}$ by joining their roots to a new point taken as the root of $T^{(1)} \vee T^{(2)}$ (Figure 1.3). Then

$$\text{br}(T^{(1)} \vee T^{(2)}) = \text{br } T^{(1)} \vee \text{br } T^{(2)}$$

since water can flow in the join $T^{(1)} \vee T^{(2)}$ iff water can flow in one of the trees.

![Figure 1.3. Joining two trees.](image)

We will denote by $|x|$ the distance of a vertex $x$ to the root.

**Example 1.4.** We will put two trees together such that $\text{br}(T^{(1)} \vee T^{(2)}) = 1$ but $\overline{\text{gr}}(T^{(1)} \vee T^{(2)}) > 1$. Let $n_k \uparrow \infty$. Let $T^{(1)}$ (resp., $T^{(2)}$) be a tree such that $x$ has 1 child (resp., 2 children) for $n_{2k} \leq |x| \leq n_{2k+1}$ and 2 (resp., 1) otherwise. If $n_k$ increases sufficiently
§2. Electric Current

rapidly, then $\text{br } T^{(1)} = \text{br } T^{(2)} = 1$, so $\text{br } (T^{(1)} \lor T^{(2)}) = 1$. But if $n_k$ increases sufficiently rapidly, then $\text{gr } (T^{(1)} \lor T^{(2)}) = \sqrt{2}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tree.png}
\caption{A schematic representation of a tree with branching number 1 and growth rate $\sqrt{2}$.}
\end{figure}

\begin{exercise}
Verify that if $\langle n_k \rangle$ increases sufficiently rapidly, then $\text{gr } (T^{(1)} \lor T^{(2)}) = \sqrt{2}$. Furthermore, show that the set of possible values of $\text{gr } (T^{(1)} \lor T^{(2)})$ over all sequences $\langle n_k \rangle$ is $[\sqrt{2}, 2]$.
\end{exercise}

While $\text{gr } T$ is easy to compute, $\text{br } T$ may not be. Nevertheless, it is the branching number which is much more important. Fortunately, Furstenberg (1967) gave a useful condition sufficient for equality in (1.1): Given a vertex $x$ in $T$, let $T^x$ denote the subtree of $T$ formed by the descendants of $x$. This tree is rooted at $x$.

\begin{theorem}
If for all vertices $x \in T$, there is an isomorphism of $T^x$ as a rooted tree to a subtree of $T$ rooted at $o$, then $\text{br } T = \text{gr } T$.
\end{theorem}

We call trees satisfying the hypothesis of this theorem subperiodic; actually, we will later broaden slightly this definition. As we will see, subperiodic trees arise naturally, which accounts for the importance of Furstenberg’s theorem.

§1.2. Electric Current.

We can ask another flow question on trees, namely: If $\lambda^{-n}$ is the conductance of edges at distance $n$ from the root of $T$ and a battery is connected between the root and “infinity”, will current flow? Of course, what we mean is that we establish a unit potential between the root and level $N$ of $T$, let $N \rightarrow \infty$, and see whether the limiting current is positive. If so, the tree is said to have positive effective conductance and finite effective resistance. (All electrical terms will be carefully explained in Chapter 2.)
Example 1.5. Consider the binary tree. By symmetry, all the vertices at a given distance from $o$ have the same potential, so they may be identified (“soldered” together) without changing any voltages or currents. This gives a new graph whose vertices may be identified with $\mathbb{N}$, while there are $2^n$ edges joining $n-1$ to $n$. These edges are in parallel, so they may be replaced by a single edge whose conductance is their sum, $(2/\lambda)^n$. Now we have edges in series, so the effective resistance is the sum of the edge resistances, $\sum_n (\lambda/2)^n$. This is finite iff $\lambda < 2$. Thus, current flows in the infinite binary tree iff $\lambda < 2$. Note the slight difference to water flow: when $\lambda = 2$, water can still flow on the binary tree.

In general, there will be a critical value of $\lambda$ below which current flows and above which it does not. It turns out that this critical value is the same as that for water flow (Lyons, 1990):

**Theorem 1.6.** If $\lambda < \text{br} \ T$, then electrical current flows, but if $\lambda > \text{br} \ T$, then it does not.

§1.3. Random Walks.

There is a well-known and easily established correspondence between electrical networks and random walks that holds for all graphs. Namely, given a finite connected graph $G$ with conductances assigned to the edges, we consider the random walk that can go from a vertex only to an adjacent vertex and whose transition probabilities from a vertex are proportional to the conductances along the edges to be taken. That is, if $x$ is a vertex with neighbors $y_1, \ldots, y_d$ and the conductance of the edge $(x, y_i)$ is $c_i$, then $p(x, y_j) = c_j / \sum_{i=1}^d c_i$. Now consider two fixed vertices $a_0$ and $a_1$ of $G$. Suppose that a battery is connected across them so that the voltage at $a_i$ equals $i$ ($i = 0, 1$). Then certain currents will flow along the edges and establish certain voltages at the other vertices in accordance with Kirchhoff’s law and Ohm’s law. The following proposition provides the basic connection between random walks and electrical networks:

**Proposition 1.7.** For any vertex $x$, the voltage at $x$ equals the probability that the corresponding random walk visits $a_1$ before it visits $a_0$ when it starts at $x$.

In fact, the proof of this proposition is simple: there is a discrete Laplacian (a difference operator) for which both the voltage and the probability mentioned are harmonic functions of $x$. The two functions clearly have the same values at $a_i$ (the boundary points) and the uniqueness principle holds for this Laplacian, whence the functions agree at all vertices $x$. This is developed in detail in Section 2.1. A superb elementary exposition of this correspondence is given by Doyle and Snell (1984).
What does this say about our trees? Given $N$, identify all the vertices of level $N$, i.e., $T_N$, to one vertex, $a_1$ (see Figure 1.5). Use the root as $a_0$. Then according to Proposition 1.7, the voltage at $x$ is the probability that the random walk visits level $N$ before it visits the root when it starts from $x$. When $N \to \infty$, the limiting voltages are all 0 iff the limiting probabilities are all 0, which is the same thing as saying that on the infinite tree, the probability of visiting the root from any vertex is 1, i.e., the random walk is recurrent. Since no current flows across edges whose endpoints have the same voltage, we see that no electrical current flows iff the random walk is recurrent.

![Figure 1.5. Identifying a level to a vertex, $a_1$.](image)

![Figure 1.6. The relative weights at a vertex. The tree is growing upwards.](image)

Now when the conductances decrease by a factor of $\lambda$ as the distance increases, the relative weights at a vertex other than the root are as shown in Figure 1.6. That is, the edge leading back toward the root is $\lambda$ times as likely to be taken as each other edge. Denoting the dependence of the random walk on the parameter $\lambda$ by $\text{RW}_\lambda$, we may translate Theorem 1.6 into the following theorem (Lyons, 1990):

**Theorem 2.23.** If $\lambda < \text{br}T$, then $\text{RW}_\lambda$ is transient, while if $\lambda > \text{br}T$, then $\text{RW}_\lambda$ is recurrent.

Is this intuitive? Consider a vertex other than the root with, say, $d$ children. If we consider only the distance from $o$, which increases or decreases at each step of the
random walk, a balance between increasing and decreasing occurs when $\lambda = d$. If $d$ were constant, it is easy to see that indeed $d$ would be the critical value separating transience from recurrence. What Theorem 2.23 says is that this same heuristic can be used in the general case, provided we substitute the “average” $\text{br} T$ for $d$.

We will also see how to use electrical networks to prove Pólya’s wonderful theorem that simple random walk on the hypercubic lattice $\mathbb{Z}^d$ is recurrent for $d \leq 2$ and transient for $d \geq 3$.

§1.4. Percolation.

Suppose that we remove edges at random from $T$. To be specific, keep each edge with some fixed probability $p$ and make these decisions independently for different edges. This random process is called percolation. By Kolmogorov’s 0-1 law, the probability that an infinite connected component remains in the tree is either 0 or 1. On the other hand, this probability is monotonic in $p$, whence there is a critical value $p_c(T)$ where it changes from 0 to 1. It is also clear that the “bigger” the tree, the more likely it is that there will be an infinite component for a given $p$. That is, the “bigger” the tree, the smaller the critical value $p_c$. Thus, $p_c$ is vaguely inversely related to a notion of average branching number. Actually, this vague heuristic is precise (Lyons, 1990):

**Theorem 4.16.** For any tree, $p_c(T) = 1/\text{br} T$.

Let us look more closely at the intuition behind this. If a vertex $x$ has $d$ children, then the expected number of children after percolation is $dp$. If $dp$ is “usually” less than 1, one would not expect that an infinite component would remain, while if $dp$ is “usually” greater than 1, then one might guess that an infinite component would be present. Theorem 4.16 says that this intuition becomes correct when one replaces the “usual” $d$ by $\text{br} T$.

Note that there is an infinite component with probability 1 iff the component of the root is infinite with positive probability.
§1.5. Branching Processes.

Percolation on a fixed tree produces random trees by random pruning, but there is a way to grow trees randomly due to Bienaymé in 1845. Given probabilities $p_k$ adding to 1 ($k = 0, 1, 2, \ldots$), we begin with one individual and let it reproduce according to these probabilities, i.e., it has $k$ children with probability $p_k$. Each of these children (if there are any) then reproduce independently with the same law, and so on forever or until some generation goes extinct. The family trees produced by such a process are called (Bienaymé)-Galton-Watson trees. A fundamental theorem in the subject is that extinction is a.s. iff $m \leq 1$ and $p_1 < 1$, where $m := \sum_k k p_k$ is the mean number of offspring. This provides further justification for the intuition we sketched behind Theorem 4.16. It also raises a natural question: Given that a Galton-Watson tree is nonextinct (infinite), what is its branching number? All the intuition suggests that it is $m$ a.s., and indeed it is. This was first proved by Hawkes (1981). But here is the idea of a very simple proof (Lyons, 1990).

According to Theorem 4.16, to determine $\text{br} \ T$, we may determine $p_c(T)$. Thus, let $T$ grow according to a Galton-Watson process, then perform percolation on $T$, i.e., keep edges with probability $p$. We are interested in the component of the root. Looked at as a random tree in itself, this component appears simply as some other Galton-Watson tree; its mean is $mp$ by independence of the growing and the “pruning” (percolation). Hence, the component of the root is infinite w.p.p. iff $mp > 1$. This says that $p_c = 1/m$ a.s. on nonextinction, i.e., $\text{br} \ T = m$.

Let $Z_n$ be the size of the $n$th generation in a Galton-Watson process. How quickly does $Z_n$ grow? It will be easy to calculate that $E[Z_n] = m^n$. Moreover, a martingale argument will show that the limit $W := \lim_{n \to \infty} Z_n/m^n$ always exists (and is finite). When $1 < m < \infty$, do we have that $W > 0$ on the event of nonextinction? The answer is “yes”, under a very mild hypothesis:

The Kesten-Stigum Theorem (1966). The following are equivalent when $1 < m < \infty$:

(i) $W > 0$ a.s. on the event of nonextinction;
(ii) $\sum_{k=1}^{\infty} p_k k \log k < \infty$.

This will be shown in Section 11.2. Although condition (ii) appears technical, we will enjoy a conceptual proof of the theorem that uses only the crudest estimates.
§1.6. Random Spanning Trees.

This fertile and fascinating field is one of the oldest areas to be studied in this book, but one of the newest to be explored in depth. A **spanning tree** of a (connected) graph is a subgraph that is connected, contains every vertex of the whole graph, and contains no cycle: see Figure 1.7 for an example. The subject of random spanning trees of a graph goes back to Kirchhoff (1847), who showed its relation to electrical networks. (Actually, Kirchhoff did not think probabilistically, but, rather, he considered quotients of the number of spanning trees with a certain property divided by the total number of spanning trees.)

One of these relations gives the probability that a uniformly chosen spanning tree will contain a given edge in terms of electrical current in the graph.

![Figure 1.7. A spanning tree in a graph, where the edges of the graph not in the tree are dashed.](image)

Let’s begin with a very simple finite graph. Namely, consider the ladder graph of Figure 1.8. Among all spanning trees of this graph, what proportion contain the bottom rung (edge)? In other words, if we were to choose at random a spanning tree, what is the chance that it would contain the bottom rung? We have illustrated the entire probability spaces for the smallest ladder graphs in Figure 1.9.

As shown, the probabilities in these cases are $1/1$, $3/4$, and $11/15$. The next one is $41/56$. Do you see any pattern? One thing that is fairly evident is that these numbers are decreasing, but hardly changing. It turns out that they come from every other term of the continued fraction expansion of $\sqrt{3} - 1 = 0.73^{+}$ and, in particular, converge to $\sqrt{3} - 1$. In the limit, then, the probability of using the bottom rung is $\sqrt{3} - 1$ and, even before
taking the limit, this gives an excellent approximation to the probability. How can we easily calculate such numbers? In this case, there is a rather easy recursion to set up and solve, but we will use this example to illustrate the more general theorem of Kirchhoff that we mentioned above. In fact, Kirchhoff’s theorem will show us why these probabilities are decreasing even before we calculate them.

Suppose that each edge of our graph is an electric conductor of unit conductance. Hook up a battery between the endpoints of any edge \( e \), say the bottom rung (Figure 1.10). Kirchhoff (1847) showed that the proportion of current that flows directly along \( e \) is then equal to the probability that \( e \) belongs to a randomly chosen spanning tree!

Coming back to the ladder graph and its bottom rung, \( e \), we see that current flows in two ways: some flows directly across \( e \) and some flows through the rest of the network.
It is intuitively clear (and justified by Rayleigh’s monotonicity principle) that the higher the ladder, the greater the effective conductance of the ladder minus the bottom rung, hence, by Kirchhoff’s theorem, the less the chance that a random spanning tree contains the bottom rung. This confirms our observations.

It turns out that generating spanning trees at random according to the uniform measure is of interest to computer scientists, who have developed various algorithms over the years for random generation of spanning trees. In particular, this is closely connected to generating a random state from any Markov chain. See Propp and Wilson (1998) for more on this issue.

Early algorithms for generating a random spanning tree used the Matrix-Tree Theorem, which counts the number of spanning trees in a graph via a determinant. A better algorithm than these early ones, especially for probabilists, was introduced by Aldous (1990) and Broder (1989). It says that if you start a simple random walk at any vertex of a finite (connected) graph $G$ and draw every edge it traverses except when it would complete a cycle (i.e., except when it arrives at a previously-visited vertex), then when no more edges can be added without creating a cycle, what will be drawn is a uniformly chosen spanning tree of $G$. (To be precise: if $X_n$ ($n \geq 0$) is the path* of the random walk, then the associated spanning tree is the set of edges $\{[X_n, X_{n+1}] ; X_{n+1} \notin \{X_0, X_1, \ldots, X_n\}\}$.)

This beautiful algorithm is quite efficient and useful for theoretical analysis, yet Wilson (1996) found an even better one that we’ll describe in Section 8.1.

Return for a moment to the ladder graphs. We saw that as the height of the ladder tends to infinity, there is a limiting probability that the bottom rung of the ladder graph

---

*In graph theory, a “path” is necessarily self-avoiding. What we call a “path” is called in graph theory a “walk”. However, to avoid confusion with random walks, we do not adopt that terminology. When a path does not pass through any vertex (resp., edge) more than once, we will call it vertex simple (resp., edge simple).
belongs to a uniform spanning tree. This suggests looking at uniform spanning trees on general infinite graphs. So suppose that $G$ is an infinite graph. Let $G_n$ be finite subgraphs with $G_1 \subset G_2 \subset G_3 \subset \cdots$ and $\bigcup G_n = G$. Pemantle (1991) showed that the weak limit of the uniform spanning tree measures on $G_n$ exists, as conjectured by Lyons. (In other words, if $\mu_n$ denotes the uniform spanning tree measure on $G_n$ and $B, B'$ are finite sets of edges, then $\lim_n \mu_n(B \subset T, B' \cap T = \emptyset)$ exists, where $T$ denotes a random spanning tree.) This limit is now called the \textbf{free uniform spanning forest*} on $G$, denoted FUSF. Considerations of electrical networks play the dominant role in Pemantle’s proof. Pemantle (1991) discovered the amazing fact that on $\mathbb{Z}^d$, the uniform spanning forest is a single tree a.s. if $d \leq 4$; but when $d \geq 5$, there are infinitely many trees a.s.!

\section*{1.7. Hausdorff Dimension.}

Consider again any finite connected graph with two distinguished vertices $a$ and $z$. This time, the edges $e$ have assigned positive numbers $c(e)$ that represent the maximum amount of water that can flow through the edge (in either direction). How much water can flow into $a$ and out of $z$? Consider any set $\Pi$ of edges that separates $a$ from $z$, i.e., the removal of all edges in $\Pi$ would leave $a$ and $z$ in different components. Such a set $\Pi$ is called a \textbf{cutset}. Since all the water must flow through $\Pi$, an upper bound for the maximum flow from $a$ to $z$ is $\sum_{e \in \Pi} c(e)$. The beautiful Max-Flow Min-Cut Theorem of Ford and Fulkerson (1962) says that these are the only constraints: the maximum flow equals $\min_{\Pi}$ a cutset $\sum_{e \in \Pi} c(e)$.

Applying this theorem to our tree situation where the amount of water that can flow through an edge at distance $n$ from the root is limited to $\lambda^{-n}$, we see that the maximum flow from the root to infinity is

$$\inf \left\{ \sum_{x \in \Pi} \lambda^{-|x|}; \Pi \text{ cuts the root from infinity} \right\} .$$

Here, we identify a set of vertices $\Pi$ with their preceding edges when considering cutsets.

By analogy with the leaves of a finite tree, we call the set of rays of $T$ the \textbf{boundary} (at infinity) of $T$, denoted $\partial T$. (It does not include any leaves of $T$.) Now there is a natural metric on $\partial T$: if $\xi, \eta \in \partial T$ have exactly $n$ edges in common, define their distance to be

* In graph theory, “spanning forest” usually means a maximal subgraph without cycles, i.e., a spanning tree in each connected component. We mean, instead, a subgraph without cycles that contains every vertex.
\[ d(\xi, \eta) := e^{-n}. \] Thus, if \( x \in T \) has more than one child with infinitely many descendants, the set of rays going through \( x \),

\[ B_x := \{ \xi \in \partial T; \xi_{[x]} = x \}, \]

has diameter \( \text{diam} B_x = e^{-|x|} \). We call a collection \( \mathcal{C} \) of subsets of \( \partial T \) a **cover** if

\[ \bigcup_{B \in \mathcal{C}} B = \partial T. \]

Note that \( \Pi \) is a cutset iff \( \{ B_x; x \in \Pi \} \) is a cover. The **Hausdorff dimension** of \( \partial T \) is defined to be

\[ \dim \partial T := \sup \left\{ \alpha; \inf_{\mathcal{C} \text{ a countable cover}} \sum_{B \in \mathcal{C}} (\text{diam } B)^\alpha > 0 \right\}. \]

This is, in fact, already familiar to us, since

\[ \text{br } T = \sup \{ \lambda; \text{ water can flow through pipe capacities } \lambda^{-|x|} \} \]

\[ = \sup \left\{ \lambda; \inf_{\Pi \text{ a cutset}} \sum_{x \in \Pi} \lambda^{-|x|} > 0 \right\} \]

\[ = \exp \sup \left\{ \alpha; \inf_{\Pi \text{ a cutset}} \sum_{x \in \Pi} e^{-\alpha|x|} > 0 \right\} \]

\[ = \exp \sup \left\{ \alpha; \inf_{\mathcal{C} \text{ a cover}} \sum_{B \in \mathcal{C}} (\text{diam } B)^\alpha > 0 \right\} \]

\[ = \exp \dim \partial T. \]

### §1.8. Capacity.

An important tool in analyzing electrical networks is that of energy. Thomson’s principle says that, given a finite graph and two distinguished vertices \( a, z \), the unit current flow from \( a \) to \( z \) is the unit flow from \( a \) to \( z \) that minimizes energy, where, if the conductances are \( c(e) \), the **energy** of a flow \( \theta \) is defined to be \( \sum_e \text{ an edge } \theta(e)^2 / c(e) \). (It turns out that the energy of the unit current flow is equal to the effective resistance from \( a \) to \( z \).) Thus,

\[ \text{electrical current flows from the root of an infinite tree} \]

\[ \iff \]

\[ \text{there is a flow with finite energy.} \]

§8. Capacity

Now on a tree, a unit flow can be identified with a function $\theta$ on the vertices of $T$ that is 1 at the root and has the property that for all vertices $x$,

$$\theta(x) = \sum_i \theta(y_i),$$

where $y_i$ are the children of $x$. The energy of a flow is then

$$\sum_{x \in T} \theta(x)^2 \lambda^{\|x\|},$$

whence

$$\text{br}_T = \sup \left\{ \lambda ; \text{there exists a unit flow } \theta \sum_{x \in T} \theta(x)^2 \lambda^{\|x\|} < \infty \right\}. \quad (1.3)$$

We can also identify unit flows on $T$ with Borel probability measures $\mu$ on $\partial T$ via

$$\mu(B_x) = \theta(x).$$

A bit of algebra shows that (1.3) is equivalent to

$$\text{br}_T = \exp \sup \left\{ \alpha ; \exists \text{ a probability measure } \mu \text{ on } \partial T \int \int \frac{d\mu(\xi)d\mu(\eta)}{d(\xi,\eta)^{\alpha}} < \infty \right\}.$$

For $\alpha > 0$, define the $\alpha$-capacity to be the reciprocal of the minimum $e^\alpha$-energy:

$$\text{cap}_\alpha(\partial T)^{-1} := \inf \left\{ \int \int \frac{d\mu(\xi)d\mu(\eta)}{d(\xi,\eta)^{\alpha}} ; \text{ } \mu \text{ a probability measure on } \partial T \right\}.$$

Then statement (1.2) says that for $\alpha > 0$,

random walk with parameter $\lambda = e^\alpha$ is transient $\iff$ $\text{cap}_\alpha(\partial T) > 0. \quad (1.4)$

It follows from Theorem 2.23 that

the critical value of $\alpha$ for positivity of $\text{cap}_\alpha(\partial T)$ is $\dim \partial T$. \quad (1.5)

A refinement of Theorem 4.16 is (Lyons, 1992):

**Theorem 14.3.** For $\alpha > 0$, percolation with parameter $p = e^{-\alpha}$ yields an infinite component a.s. iff $\text{cap}_\alpha(\partial T) > 0$. Moreover,

$$\text{cap}_\alpha(\partial T) \leq P[\text{the component of the root is infinite}] \leq 2 \text{cap}_\alpha(\partial T).$$

When $T$ is spherically symmetric and $p = e^{-\alpha}$, we have (Exercise 14.1)
\[
cap_{\alpha}(\partial T) = \left(1 + (1 - p) \sum_{n=1}^{\infty} \frac{1}{p^n |T_n|}\right)^{-1}.
\]

The case of the first part of this theorem where all the degrees are uniformly bounded was shown first by Fan (1989, 1990).

One way to use this theorem is to combine it with (1.4); this allows us to translate problems freely between the domains of random walks and percolation (Lyons, 1992). The theorem actually holds in a much wider context (on trees) than that mentioned here: Fan allowed the probabilities $p$ to vary depending on the generation and Lyons allowed the probabilities as well as the tree to be completely arbitrary.

§1.9. Embedding Trees into Euclidean Space.

The results described above, especially those concerning percolation, can be translated to give results on closed sets in Euclidean space. We will describe only the simplest such correspondence here, which was the one that was part of Furstenberg’s motivation in 1970. Namely, for a closed nonempty set $E \subseteq [0, 1]$ and for any integer $b \geq 2$, consider the system of $b$-adic subintervals of $[0, 1]$. Those whose intersection with $E$ is non-empty will form the vertices of the associated tree. Two such intervals are connected by an edge iff one contains the other and the ratio of their lengths is $b$. The root of this tree is $[0, 1]$. We denote the tree by $T_b(E)$. Were it not for the fact that certain numbers have two representations in base $b$, we could identify $\partial T_b(E)$ with $E$. Note that such an identification would be Hölder continuous in one direction (only).

\[
\begin{array}{cccccc}
0 & \hline & \hline & \hline & \hline & 1 \\
\vdots & & & & & \\
\end{array}
\]

\textbf{Figure 1.11.} In this case $b = 4.$
Hausdorff dimension is defined for subsets of $[0, 1]$ just as it was for $\partial T$:

$$\dim E := \sup \left\{ \alpha \colon \inf_{C \text{ a cover of } E} \sum_{B \in C} (\text{diam } B)^\alpha > 0 \right\},$$

where $\text{diam } B$ denotes the (Euclidean) diameter of $E$. Covers of $\partial T_b(E)$ by sets of the form $B_x$ correspond to covers of $E$ by $b$-adic intervals, but of diameter $b^{-|x|}$, rather than $e^{-|x|}$. It is easy to show that restricting to such covers does not change the computation of Hausdorff dimension, whence we may conclude (compare the calculation at the end of Section 1.7) that

$$\dim E = \frac{\dim \partial T_b(E)}{\log b} = \log_b (\text{br } T_b(E)).$$

For example, the Hausdorff dimension of the Cantor middle-thirds set is $\log 2/\log 3$, which we see by using the base $b = 3$. It is interesting that if we use a different base, we will still have $\text{br } T_b(E) = 2$ when $E$ is the Cantor set.

Capacity is also defined as it was on the boundary of a tree:

$$(\text{cap}_\alpha E)^{-1} := \inf \left\{ \int \int \frac{d\mu(x) d\mu(y)}{|x - y|^{\alpha}} \colon \mu \text{ a probability measure on } E \right\}.$$

It was shown by Benjamini and Peres (1992) (see Section 14.3) that

$$\frac{\text{cap}_\alpha E}{3} \leq \text{cap}_\alpha \log_b (\partial T_b(E)) \leq b \text{ cap}_\alpha E. \quad (1.6)$$

This means that the percolation criterion Theorem 14.3 can be used in Euclidean space. In fact, inequalities such as (1.6) also hold for more general kernels than distance to a power and for higher-dimensional Euclidean spaces. Nice applications to the path of Brownian motion were found by Peres (1996) by replacing the path by an “intersection-equivalent” random fractal that is much easier to analyze, being an embedding of a Galton-Watson tree. (This is planned for future inclusion in the book.)

Also, the fact (1.5) translates into the analogous statement about subsets $E \subseteq [0, 1]$; this is a classical theorem of Frostman (1935).
Chapter 2

Random Walks and Electric Networks

There are profound and detailed connections between potential theory and probability; see, e.g., Chapter II of Bass (1995) or Doob (1984). We will look at a simple discrete version of this connection that is very useful. A superb elementary introduction to the ideas of the first five sections of this chapter is given by Doyle and Snell (1984).

§2.1. Circuit Basics and Harmonic Functions.

Our principal interest in this chapter centers around transience and recurrence of irreducible Markov chains. If the chain starts at a state \( x \), then we want to know whether the chance that it ever visits a state \( a \) is 1 or not.

In fact, we are interested only in reversible Markov chains, where we call a Markov chain reversible if there is a positive function \( x \mapsto \pi(x) \) on the state space such that the transition probabilities satisfy \( \pi(x)p_{xy} = \pi(y)p_{yx} \) for all pairs of states \( x, y \). (Such a function \( \pi(\cdot) \) will then provide a stationary measure: see Exercise 2.1. Note that \( \pi(\cdot) \) is not generally a probability measure.) In this case, make a graph \( G \) by taking the states of the Markov chain for vertices and joining two vertices \( x, y \) by an edge when \( p_{xy} > 0 \). Assign weight

\[
c(x, y) := \pi(x)p_{xy}
\]  

(2.1)

to that edge; note that the condition of reversibility ensures that this weight is the same no matter in what order we take the endpoints of the edge. With this network in hand, the Markov chain may be described as a random walk on \( G \): when the walk is at a vertex \( x \), it chooses randomly among the vertices adjacent to \( x \) with transition probabilities proportional to the weights of the edges. Conversely, every connected graph with weights on the edges such that the sum of the weights incident to every vertex is finite gives rise to a random walk with transition probabilities proportional to the weights. Such a random walk is an irreducible reversible Markov chain: define \( \pi(x) \) to be the sum of the weights incident to \( x \).*

* Suppose that we consider an edge \( e \) of \( G \) to have length \( c(e)^{-1} \). Run a Brownian motion on \( G \) and
The most well-known example is gambler’s ruin. A gambler needs $n$ but has only $k$ ($1 \leq k \leq n-1$). He plays games that give him chance $p$ of winning $1$ and $q := 1-p$ of losing $1$ each time. When his fortune is either $n$ or 0, he stops. What is his chance of ruin (i.e., reaching 0 before $n$)? We will answer this in Example 2.4 by using the following weighted graph. The vertices are $\{0, 1, 2, \ldots, n\}$, the edges are between consecutive integers, and the weights are $c(i, i+1) = c(i+1, i) = (p/q)^i$.

**Exercise 2.1.**

This exercise contains some background information and facts that we will use about reversible Markov chains.

(a) Show that if a Markov chain is reversible, then $\forall x_1, x_2, \ldots, x_n,$

\[
\pi(x_1) \prod_{i=1}^{n-1} p_{x_i, x_{i+1}} = \pi(x_n) \prod_{i=1}^{n-1} p_{x_{n-i}, x_{n-i}} ,
\]

whence $\prod_{i=1}^{n-1} p_{x_i, x_{i+1}} = \prod_{i=1}^{n-1} p_{x_{n-i}, x_{n-i}}$ if $x_1 = x_n$. This last equation also characterizes reversibility.

(b) Let $\langle X_n \rangle$ be a random walk on $G$ and let $x$ and $y$ be two vertices in $G$. Let $P$ be a path from $x$ to $y$ and $P'$ its reversal, a path from $y$ to $x$. Show that

\[
P_x [\langle X_n \rangle; n \leq \tau_y] = P \mid \tau_y < \tau_x^+ = P_y [\langle X_n \rangle; n \leq \tau_x] = P' \mid \tau_x < \tau_y^+ ,
\]

where $\tau_w$ denotes the first time the random walk visits $w$, $\tau_w^+$ denotes the first time after 0 that the random walk visits $w$, and $P_u$ denotes the law of random walk started at $u$. In words, paths between two states that don’t return to the starting point and stop at the first visit to the endpoint have the same distribution in both directions of time.

(c) Consider a random walk on $G$ that is either transient or is stopped on the first visit to a set of vertices $Z$. Let $G(x, y)$ be the expected number of visits to $y$ for a random walk started at $x$; if the walk is stopped at $Z$, we count only those visits that occur strictly before visiting $Z$. Show that for every pair of vertices $x$ and $y$,

\[
\pi(x) G(x, y) = \pi(y) G(y, x) .
\]

(d) Show that random walk on a connected weighted graph $G$ is positive recurrent (i.e., has a stationary probability distribution) iff $\sum_{x, y} c(x, y) < \infty$, in which case the stationary probability distribution is proportional to $\pi(\bullet)$. Show that if the random walk is not positive recurrent, then $\pi(\bullet)$ is a stationary infinite measure.

---

We begin by studying random walks on finite networks. Let $G$ be a finite connected network, $x$ a vertex of $G$, and $A$, $Z$ disjoint subsets of vertices of $G$. Let $\tau_A$ be the first time that the random walk visits (“hits”) some vertex in $A$; if the random walk happens to start in $A$, then this is 0. Occasionally, we will use $\tau_A^+$, which is the first time after 0 that the walk visits $A$; this is different from $\tau_A$ only when the walk starts in $A$. Usually $A$ and $Z$ will be singletons. Often, all the edge weights are equal; we call this case simple random walk.

Consider the probability that the random walk visits $A$ before it visits $Z$ as a function of its starting point $x$:

$$F(x) := \Pr_x[\tau_A < \tau_Z].$$

Recall that $\upharpoonright$ indicates the restriction of a function to a set. Clearly $F\upharpoonright A \equiv 1$, $F\upharpoonright Z \equiv 0$, and for $x \notin A \cup Z$,

$$F(x) = \sum_y \Pr_x[\text{first step is to } y] \Pr_x[\tau_A < \tau_Z \mid \text{first step is to } y]$$

$$= \sum_{x \sim y} p_{xy} F(y) = \frac{1}{\pi(x)} \sum_{x \sim y} c(x, y) F(y),$$

where $x \sim y$ indicates that $x$, $y$ are adjacent in $G$. In the special case of simple random walk, this equation becomes

$$F(x) = \frac{1}{\deg x} \sum_{x \sim y} F(y),$$

where $\deg x$ is the degree of $x$, i.e., the number of edges incident to $x$. That is, $F(x)$ is the average of the values of $F$ at the neighbors of $x$. In general, this is still true, but the average is taken with weights.

We say that a function $f$ is harmonic at $x$ when

$$f(x) = \frac{1}{\pi(x)} \sum_{x \sim y} c(x, y) f(y).$$

If $f$ is harmonic at each point of a set $W$, then we say that $f$ is harmonic on $W$. Harmonic functions satisfy a maximum principle: For $W \subseteq V(G)$, write $\overline{W}$ for the set of vertices that are either in $W$ or are adjacent to some vertex in $W$.

**Maximum Principle.** Let $G$ be a finite or infinite network. If $H \subseteq G$, $H$ is connected, $f : V(G) \to \mathbb{R}$, $f$ is harmonic on $V(H)$, and the maximum of $f$ on $V(H)$ is achieved and is equal to the supremum of $f$ on $G$, then $f\upharpoonright \overline{V(H)} \equiv \max f$.

**Proof.** Let $K := \{ y \in \overline{V(H)} ; \ f(y) = \max f \}$. Note that if $x \in V(H)$, $x \sim y$, and $f(x) = \max f$, then $f(y) = \max f$ by harmonicity of $f$ at $x$. Thus, $\overline{K} \cap \overline{V(H)} = K$. Since $H$ is connected and $K \neq \emptyset$, it follows that $K = \overline{V(H)}$. $\blacksquare$
This leads to the

**Uniqueness Principle.** Let \( G = (V, E) \) be a finite or infinite connected network. Let \( W \) be a finite proper subset of \( V \). If \( f, g : V \to \mathbb{R} \), \( f, g \) are harmonic on \( W \), and \( f|\{V \setminus W\} = g|\{V \setminus W\} \), then \( f = g \).

**Proof.** Let \( h := f - g \). We claim that \( h \leq 0 \). This suffices to establish the corollary since then \( h \geq 0 \) by symmetry, whence \( h = 0 \).

Now \( h = 0 \) off \( W \), so if \( h \nleq 0 \) on \( W \), then \( h \) is positive somewhere on \( W \), whence \( \max h|W = \max h \). Let \( H \) be a connected component of \( (W, E \cap (W \times W)) \) where \( h \) achieves its maximum. According to the maximum principle, \( h \) is a positive constant on \( V(H) \). In particular, \( h > 0 \) on the non-empty set \( V(H) \setminus V(H) \). However, \( V(H) \setminus V(H) \subseteq V \setminus W \), whence \( h = 0 \) on \( V(H) \setminus V(H) \). This is a contradiction. \( \blacktriangleleft \)

Thus, the harmonicity of the function \( x \mapsto P_x[\tau_A < \tau_Z] \) (together with its values where it is not harmonic) characterizes it.

If \( f, f_1, \) and \( f_2 \) are harmonic on \( W \) and \( a_1, a_2 \in \mathbb{R} \) with \( f = a_1 f_1 + a_2 f_2 \) on \( V \setminus W \), then \( f = a_1 f_1 + a_2 f_2 \) everywhere by the uniqueness principle. This is one form of the **superposition principle**.

**Existence Principle.** Let \( G = (V, E) \) be a finite or infinite network. If \( W \subsetneq V \) and \( f_0 : V \setminus W \to \mathbb{R} \) is bounded, then \( \exists f : V \to \mathbb{R} \) such that \( f|\{V \setminus W\} = f_0 \) and \( f \) is harmonic on \( W \).

**Proof.** For any starting point \( x \) of the network random walk, let \( X \) be the first vertex in \( V \setminus W \) visited by the random walk if \( V \setminus W \) is indeed visited. Let \( Y := f_0(X) \) when \( V \setminus W \) is visited and \( Y := 0 \) otherwise. It is easily checked that \( f(x) := E_x[Y] \) works in the same way as we discovered that the function \( F \) of (2.2) is harmonic. \( \blacktriangleleft \)

This is the solution to the so-called Dirichlet problem. The function \( F \) of (2.2) is the particular case \( W = V \setminus (A \cup Z) \), \( f_0|A \equiv 1 \), and \( f_0|Z \equiv 0 \).

In fact, for finite networks, we could have immediately deduced existence from uniqueness: The Dirichlet problem on a finite network consists of a finite number of linear equations, one for each vertex in \( W \). Since the number of unknowns is equal to the number of equations, the uniqueness principle implies the existence principle.

In order to study the solution to the Dirichlet problem, especially for a sequence of subgraphs of an infinite graph, we will discover that electrical networks are useful. Electrical networks, of course, have a physical meaning whose intuition is useful to us, but also they can be used as a rigorous mathematical tool.
Mathematically, an electrical network is just a weighted graph. But now we call the weights of the edges conductances; their reciprocals are called resistances. (Note that later, we will encounter effective conductances and resistances; these are not the same.) We hook up a battery or batteries (this is just intuition) between $A$ and $Z$ so that the voltage at every vertex in $A$ is 1 and in $Z$ is 0 (more generally, so that the voltages on $V \setminus W$ are given by $f_0$). (Sometimes, voltages are called potentials or potential differences.) Voltages $v$ are then established at every vertex and current $i$ runs through the edges. These functions are implicitly defined and uniquely determined on finite networks, as we will see, by two “laws”:

**Ohm’s Law**: If $x \sim y$, the current flowing from $x$ to $y$ satisfies
\[ v(x) - v(y) = i(x, y)r(x, y). \]

**Kirchhoff’s Node Law**: The sum of all currents flowing out of a given vertex is 0, provided the vertex is not connected to a battery.

Physically, Ohm’s law, which is usually stated as $v = ir$ in engineering, is an empirical statement about linear response to voltage differences—certain components obey this law over a wide range of voltage differences. Notice also that current flows in the direction of decreasing voltage: $i(x, y) > 0$ iff $v(x) > v(y)$. Kirchhoff’s node law expresses the fact that charge does not build up at a node (current being the passage rate of charge per unit time). If we add wires corresponding to the batteries, then the proviso in Kirchhoff’s node law is unnecessary.

Mathematically, we’ll take Ohm’s law to be the definition of current in terms of voltage. In particular, $i(x, y) = -i(y, x)$. Then Kirchhoff’s node law presents a constraint on what kind of function $v$ can be. Indeed, it determines $v$ uniquely: Current flows into $G$ at $A$ and out at $Z$. Thus, we may combine the two laws on $V \setminus (A \cup Z)$ to obtain
\[ \forall x \notin A \cup Z \quad 0 = \sum_{x \sim y} i(x, y) = \sum_{x \sim y} [v(x) - v(y)]c(x, y), \]
or
\[ v(x) = \frac{1}{\pi(x)} \sum_{x \sim y} c(x, y)v(y). \]
That is, $v(\bullet)$ is harmonic on $V \setminus (A \cup Z)$. Since $v|A \equiv 1$ and $v|Z \equiv 0$, it follows that if $G$ is finite, then $v = F$ (defined in (2.2)); in particular, we have uniqueness and existence of voltages. The voltage function is just the solution to the Dirichlet problem.
Now if we sum the differences of a function, such as the voltage \( v \), on the edges of a cycle, we get 0. Thus, by Ohm’s law, we deduce:

**Kirchhoff’s Cycle Law:** If \( x_1 \sim x_2 \sim \cdots \sim x_n \sim x_{n+1} = x_1 \) is a cycle, then

\[
\sum_{i=1}^{n} i(x_i, x_{i+1}) r(x_i, x_{i+1}) = 0.
\]

One can also deduce Ohm’s law from Kirchhoff’s two laws. A somewhat more general statement is in the following exercise.

**Exercise 2.2.**
Suppose that an antisymmetric function \( j \) (meaning that \( j(x, y) = -j(y, x) \)) on the edges of a finite connected network satisfies Kirchhoff’s cycle law and Kirchhoff’s node law in the form \( \sum_{x \sim y} j(x, y) = 0 \) for all \( x \notin W \). Show that \( j \) is the current determined by imposing voltages on \( W \) and that the voltage function is unique up to an additive constant.

**§2.2. More Probabilistic Interpretations.**

Suppose that \( A = \{a\} \) is a singleton. What is the chance that a random walk starting at \( a \) will hit \( Z \) before it returns to \( a \)? Write this as

\[
P[a \to Z] := P_a[\tau_Z < \tau_{\{a\}}^+].
\]

Impose a voltage of \( v(a) \) at \( a \) and 0 on \( Z \). Since \( v(\cdot) \) is linear in \( v(a) \) by the superposition principle, we have that \( P_x[\tau_{\{a\}} < \tau_Z] = v(x)/v(a) \), whence

\[
P[a \to Z] = \sum_x p_{ax} \left( 1 - P_x[\tau_{\{a\}} < \tau_Z] \right) = \sum_x \frac{c(a, x)}{\pi(a)} \left( 1 - \frac{v(x)}{v(a)} \right)
\]

\[
= \frac{1}{v(a)\pi(a)} \sum_x c(a, x) [v(a) - v(x)] = \frac{1}{v(a)\pi(a)} \sum_x i(a, x).
\]

In other words,

\[
v(a) = \frac{\sum_x i(a, x)}{\pi(a)P[a \to Z]}. \tag{2.3}
\]

Since \( \sum_x i(a, x) \) is the total amount of current flowing into the circuit at \( a \), we may regard the entire circuit between \( a \) and \( Z \) as a single conductor of effective conductance

\[
C_{\text{eff}} := \pi(a)P[a \to Z] =: \mathcal{C}(a \leftrightarrow Z), \tag{2.4}
\]

where the last notation indicates the dependence on $a$ and $Z$. (If we need to indicate the dependence on $G$, we will write $C(a \leftrightarrow Z; G)$.) The similarity to (2.1) can provide a good mnemonic, but the analogy should not be pushed too far. We define the effective resistance $R(a \leftrightarrow Z)$ to be its reciprocal. One answer to our question above is thus $P[a \rightarrow Z] = C(a \leftrightarrow Z)/\pi(a)$. In Sections 2.3 and 2.4, we will see some ways to compute effective conductance.

Now the number of visits to $a$ before hitting $Z$ is a geometric random variable with mean $P[a \rightarrow Z]^{-1} = \pi(a)R(a \leftrightarrow Z)$. According to (2.3), this can also be expressed as $\pi(a)v(a)$ when there is unit current flowing from $a$ to $Z$ and the voltage is 0 on $Z$. This generalizes as follows. Let $G(a, x)$ be the expected number of visits to $x$ strictly before hitting $Z$ by a random walk started at $a$. Thus, $G(a, x) = 0$ for $x \in Z$. The function $G(\bullet, \bullet)$ is the Green function for the random walk absorbed (or “killed”) on $Z$.

**Proposition 2.1.** Let $G$ be a finite connected network. When a voltage is imposed on $\{a\} \cup Z$ so that a unit current flows from $a$ to $Z$ and the voltage is 0 on $Z$, then $v(x) = G(a, x)/\pi(x)$ for all $x$.

**Proof.** We have just shown that this is true for $x \in \{a\} \cup Z$, so it suffices to establish that $G(a, x)/\pi(x)$ is harmonic elsewhere. But by Exercise 2.1, we have that $G(a, x)/\pi(x) = G(x, a)/\pi(a)$ and the harmonicity of $G(x, a)$ is established just as for the function of (2.2).

Given that we have two probabilistic interpretations of voltage, we naturally wonder whether current has a probabilistic interpretation. We might guess one by the following unrealistic but simple model of electricity: positive particles enter the circuit at $a$, they do Brownian motion on $G$ (being less likely to pass through small conductors) and, when they hit $Z$, they are removed. The net flow rate of particles across an edge would then be the current on that edge. It turns out that in our mathematical model, this is correct:

**Proposition 2.2.** Let $G$ be a finite connected network. Start a random walk at $a$ and absorb it when it first visits $Z$. For $x \sim y$, let $S_{xy}$ be the number of transitions from $x$ to $y$. Then $E[S_{xy}] = G(a, x)p_{xy}$ and $E[S_{xy} - S_{yx}] = i(x, y)$, where $i$ is the current in $G$ when a potential is applied between $a$ and $Z$ in such an amount that unit current flows in at $a$.

Note that we count a transition from $y$ to $x$ when $y \notin Z$ but $x \in Z$, although we do not count this as a visit to $x$ in computing $G(a, x)$.

**Proof.** We have

$$E[S_{xy}] = E\left[\sum_{k=0}^{\infty} 1\{X_k = x\} 1\{X_{k+1} = y\}\right] = \sum_{k=0}^{\infty} P[X_k = x, X_{k+1} = y]$$
\[\sum_{k=0}^{\infty} \mathbb{P}[X_k = x] p_{xy} = \mathbb{E}\left[\sum_{k=0}^{\infty} 1_{\{X_k = x\}}\right] p_{xy} = \mathcal{G}(a, x)p_{xy}.\]

Hence by Proposition 2.1, we have

\[
\forall x, y \quad \mathbb{E}[S_{xy} - S_{yx}] = \mathcal{G}(a, x)p_{xy} - \mathcal{G}(a, y)p_{yx}
= v(x)\pi(x)p_{xy} - v(y)\pi(y)p_{yx} = [v(x) - v(y)]c(x, y) = i(x, y). \quad \blacktriangle
\]

Effective conductance is a key quantity because of the following relationship to the question of transience and recurrence when \(G\) is infinite. Recall that for an infinite network \(G\), we assume that

\[
\forall x \sum_{x \sim y} c(x, y) < \infty, \tag{2.5}
\]

so that the associated random walk is well defined. (Of course, this is true when \(G\) is locally finite—i.e., the number of edges incident to every given vertex is finite.) We allow more than one edge between a given pair of vertices: each such edge has its own conductance. Loops are also allowed (edges with only one endpoint), but these may be ignored for our present purposes since they only delay the random walk. Strictly speaking, then, \(G\) may be a multigraph, not a graph. However, we will ignore this distinction.

\textbf{Exercise 2.3.}

In fact, we have not yet used anywhere that \(G\) has only finitely many edges. Verify that Propositions 2.1 and 2.2 are valid when the number of edges is infinite but the number of vertices is finite.

Let \((G_n)\) be any sequence of finite subgraphs of \(G\) that exhaust \(G\), i.e., \(G_n \subseteq G_{n+1}\) and \(G = \bigcup G_n\). Let \(Z_n\) be the set of vertices in \(G \setminus G_n\). (Note that if \(Z_n\) is identified to a point, the graph will have finitely many vertices but may have infinitely many edges even when loops are deleted.) Then for every \(a \in G\), the events \(\{a \to Z_n\}\) are decreasing in \(n\), so the limit \(\lim_n \mathbb{P}[a \to Z_n]\) is the probability of never returning to \(a\) in \(G\)—the escape probability from \(a\). This is positive iff the random walk on \(G\) is transient. Hence by (2.4), \(\lim_{n \to \infty} C(a \leftrightarrow Z_n) > 0\) iff the random walk on \(G\) is transient. We call \(\lim_{n \to \infty} C(a \leftrightarrow Z_n)\) the effective conductance from \(a\) to \(\infty\) in \(G\) and denote it by \(C(a \leftrightarrow \infty)\) or, if \(a\) is understood, by \(C_{\text{eff}}\). Its reciprocal, effective resistance, is denoted \(R_{\text{eff}}\). We have shown:

\textbf{Theorem 2.3.} Random walk on an infinite connected network is transient iff the effective conductance from any of its vertices to infinity is positive.
Exercise 2.4.

For a fixed vertex $a$ in $G$, show that $\lim_{n \to \infty} C(a \leftrightarrow Z_n)$ is the same for every sequence $\langle G_n \rangle$ that exhausts $G$.

Exercise 2.5.

When $G$ is finite but $A$ is not a singleton, define $C(A \leftrightarrow Z)$ to be $C(a \leftrightarrow Z)$ if all the vertices in $A$ were to be identified to a single vertex, $a$. Show that if voltages are applied at the vertices of $A \cup Z$ so that $v|A$ and $v|Z$ are constants, then $v|A - v|Z = I_{AZ} R(A \leftrightarrow Z)$, where $I_{AZ} := \sum_{x \in A} \sum_{y} i(x, y)$ is the total amount of current flowing from $A$ to $Z$.

§2.3. Network Reduction.

How do we calculate effective conductance of a network? Since we want to replace a network by an equivalent single conductor, it is natural to attempt this by replacing more and more of $G$ through simple transformations. There are, in fact, three such simple transformations, series, parallel, and star-triangle, and it turns out that they suffice to reduce all finite planar networks by a theorem of Epifanov; see Truemper (1989).

I. Series. Two resistors $r_1$ and $r_2$ in series are equivalent to a single resistor $r_1 + r_2$. In other words, if $w \in V(G) \setminus (A \cup Z)$ is a node of degree 2 with neighbors $u_1, u_2$ and we replace the edges $(u_i, w)$ by a single edge $(u_1, u_2)$ having resistance $r(u_1, w) + r(w, u_2)$, then all potentials and currents in $G \setminus \{w\}$ are unchanged and the current that flows from $u_1$ to $u_2$ equals $i(u_1, w)$.

Proof. It suffices to check that Ohm’s and Kirchhoff’s laws are satisfied on the new network for the voltages and currents given. This is easy.

Exercise 2.6.

Give two harder but instructive proofs of the series equivalence: Since voltages determine currents, it suffices to check that the voltages are as claimed on the new network $G'$. (1) Show that $v(x)$ ($x \in V(G) \setminus \{w\}$) is harmonic on $V(G') \setminus (A \cup Z)$. (2) Use the “craps principle” (Pitman (1993), p. 210) to show that $P_x[\tau_A < \tau_Z]$ is unchanged for $x \in V(G) \setminus \{w\}$.
Example 2.4. Consider simple random walk on $\mathbb{Z}$. Let $0 \leq k \leq n$. What is $\mathbb{P}_k[\tau_0 < \tau_n]$? It is the voltage at $k$ when there is a unit voltage imposed at 0 and zero voltage at $n$. If we replace the resistors in series from 0 to $k$ by a single resistor with resistance $k$ and the resistors from $k$ to $n$ by a single resistor of resistance $n - k$, then the voltage at $k$ does not change. But now this voltage is simply the probability of taking a step to 0, which is thus $(n - k)/n$.

For gambler’s ruin, rather than simple random walk, we have the conductances $c(i, i + 1) = (p/q)^i$. The replacement of edges in series by single edges now gives one edge from 0 to $k$ of resistance $\sum_{i=0}^{k-1} (q/p)^i$ and one edge from $k$ to $n$ of resistance $\sum_{i=k}^{n-1} (q/p)^i$. The probability of ruin is therefore $\sum_{i=k}^{n-1} (q/p)^i / \sum_{i=0}^{n-1} (q/p)^i = [(p/q)^{n-k} - 1]/[(p/q)^n - 1]$.

II. Parallel. Two conductors $c_1$ and $c_2$ in parallel are equivalent to one conductor $c_1 + c_2$. In other words, if two edges $e_1$ and $e_2$ that both join vertices $w_1, w_2 \in V(G)$ are replaced by a single edge $e$ joining $w_1, w_2$ of conductance $c(e) := c(e_1) + c(e_2)$, then all voltages and currents in $G \setminus \{e_1, e_2\}$ are unchanged and the current $i(e)$ equals $i(e_1) + i(e_2)$ (if $e_1$ and $e_2$ have the “same” orientations, i.e., same tail and head). The same is true for an infinite number of edges in parallel.

Proof. Check Ohm’s and Kirchhoff’s laws with $i(e) := i(e_1) + i(e_2)$.

Exercise 2.7.

Give two more proofs of the parallel equivalence as in Exercise 2.6.
Example 2.5. Suppose that each edge in the following network has equal conductance. What is $P[a \rightarrow z]$? Following the transformations indicated in the figure, we obtain $C(a \leftrightarrow z) = 7/12$, so that

$$P[a \rightarrow z] = \frac{C(a \leftrightarrow z)}{\pi(a)} = \frac{7/12}{3} = \frac{7}{36}.$$ 

Example 2.6. What is $P[a \rightarrow z]$ in the following network?

There are 2 ways to deal with the vertical edge:

(1) Remove it: by symmetry, the voltages at its endpoints are equal, whence no current flows on it.

(2) Contract it, i.e., remove it but combine its endpoints into one vertex (we could also combine the other two unlabelled vertices with each other): the voltages are the same, so they may be combined.

In either case, we get $C(a \leftrightarrow z) = 2/3$, whence $P[a \rightarrow z] = 1/3$. 
Exercise 2.8.
Let \((G, c)\) be a spherically symmetric network, meaning that if \(x\) and \(y\) are any two vertices at the same distance from \(o\), then there is an automorphism of \((G, c)\) that leaves \(o\) fixed and that takes \(x\) to \(y\). (A network automorphism is a map \(\phi : G \rightarrow G\) that is a bijection of the vertex set with itself and a bijection of the edge set with itself such that if \(x\) and \(e\) are incident, then so are \(\phi(x)\) and \(\phi(e)\) and such that \(c(e) = c(\phi(e))\) for all edges \(e\).) Let \(C_n\) be the sum of \(c(e)\) over all edges \(e\) with \(d(e^-, o) = n\) and \(d(e^+, o) = n + 1\). Show that

\[
\mathcal{R}(o \leftrightarrow \infty) = \sum_n \frac{1}{C_n},
\]
whence the network random walk on \(G\) is transient iff

\[
\sum_n \frac{1}{C_n} < \infty.
\]

III. Star-triangle. The configurations below are equivalent when

\[
\forall i \in \{1, 2, 3\} \quad c(w, u_i)c(u_{i-1}, u_{i+1}) = \gamma,
\]
where indices are taken mod 3 and

\[
\gamma := \frac{\prod_i c(w, u_i)}{\sum_i c(w, u_i)} = \frac{\sum_i r(u_{i-1}, u_{i+1})}{\prod_i r(u_{i-1}, u_{i+1})}.
\]

We won’t use this equivalence except in Example 2.7 and the exercises. This is also called the “Y-Δ” or “Wye-Delta” transformation.

Exercise 2.9.
Give at least one proof of star-triangle equivalence.

Actually, there is a fourth trivial transformation: we may prune (or add) vertices of degree 1 (and attendant edges) as well as loops.
Exercise 2.10.
Find a (finite) graph that can’t be reduced to a single edge by these four transformations.

Either of the transformations star-triangle or triangle-star can also be used to reduce the network in Example 2.6.

Example 2.7. What is $P_x[\tau_a < \tau_z]$ in the following network? Following the transformations indicated in the figure, we obtain

$$P_x[\tau_a < \tau_z] = \frac{20/33}{20/33 + 15/22} = \frac{8}{17}.$$
Since we are interested in flows on $E$, it is natural to consider that what flows one way is the negative of what flows the other. Thus, define $\ell^2(E)$ to be the space of antisymmetric functions $\theta$ on $E$ (i.e., $\theta(-e) = -\theta(e)$ for each edge $e$) with the inner product

$$(\theta, \theta') := \frac{1}{2} \sum_{e \in E} \theta(e)\theta'(e) = \sum_{e \in E_{1/2}} \theta(e)\theta'(e),$$

where $E_{1/2} \subset E$ is a set of edges containing exactly one of each pair $e, -e$. Since voltage differences across edges lead to currents, define the coboundary operator $d : \ell^2(V) \to \ell^2(E)$ by

$$(df)(e) := f(e^-) - f(e^+).$$

(Note that this is the negative of the more natural definition; since current flows from greater to lesser voltage, however, it is the more useful definition for us.) This operator is clearly linear. Conversely, given an antisymmetric function on the edges, we are interested in the net flow out of a vertex, whence we define the boundary operator $d^* : \ell^2(E) \to \ell^2(V)$ by

$$(d^*\theta)(x) := \sum_{e^- = x} \theta(e).$$

This operator is also clearly linear. We use the superscript $^*$ because these two operators are adjoints of each other:

$$\forall f \in \ell^2(V) \quad \forall \theta \in \ell^2(E) \quad (\theta, df) = (d^*\theta, f).$$

Exercise 2.11.
Prove that $d$ and $d^*$ are adjoints of each other.

One use of this notation is that the calculation left here for Exercise 2.11 need not be repeated every time it arises—and it arises a lot. Another use is the following compact forms of the network laws. Let $i$ be a current.

**Ohm’s Law:** $dv = ir$, i.e., $\forall e \in E \quad dv(e) = i(e)r(e)$.

**Kirchhoff’s Node Law:** $d^*i(x) = 0$ if no battery is connected at $x$.

It will be useful to study flows other than current in order to discover a special property of the current flow. We can imagine water flowing through a network of pipes. The amount of water flowing into the network at a vertex $a$ is $d^*\theta(a)$. Thus, we call $\theta \in \ell^2_-(E)$ a flow from $A$ to $Z$ if $d^*\theta$ is 0 off of $A$ and $Z$, is nonnegative on $A$, and nonpositive on $Z$.
The total amount flowing into the network is then $\sum_{a \in A} d^* \theta(a)$; as the next lemma shows, this is also the total amount flowing out of the network. We call

$$\text{Strength}(\theta) := \sum_{a \in A} d^* \theta(a)$$

the strength of the flow $\theta$. A flow of strength 1 is called a unit flow.

**Lemma 2.8.** Let $G$ be a finite graph and $A$ and $Z$ be two disjoint subsets of its vertices. If $\theta$ is a flow from $A$ to $Z$, then

$$\sum_{a \in A} d^* \theta(a) = -\sum_{z \in Z} d^* \theta(z).$$

**Proof.** We have

$$\sum_{x \in A} d^* \theta(x) + \sum_{x \in Z} d^* \theta(x) = \sum_{x \in A \cup Z} d^* \theta(x) = (d^* \theta, 1) = (\theta, d1) = (\theta, 0) = 0$$

since $d^* \theta(x) = 0$ for $x \not\in A \cup Z$. ▬

The following consequence will be useful in a moment.

**Lemma 2.9.** Let $G$ be a finite graph and $A$ and $Z$ be two disjoint subsets of its vertices. If $\theta$ is a flow from $A$ to $Z$ and $f|A$, $f|Z$ are constants $\alpha$ and $\zeta$, then

$$(\theta, df) = \text{Strength}(\theta)(\alpha - \zeta).$$

**Proof.** We have $$(\theta, df) = (d^* \theta, f) = \sum_{a \in A} d^* \theta(a) \alpha + \sum_{z \in Z} d^* \theta(z) \zeta.$$ Now apply Lemma 2.8. ▬

When a current $i$ flows through a resistor of resistance $r$ and voltage difference $v$, energy is dissipated at rate $P = iv = i^2 r = i^2 / c = v^2 c = v^2 / r$. We are interested in the total power (= energy per unit time) dissipated.

**Notation.** Write

$$(f, g)_h := (fh, g) = (f, gh)$$

and

$$\|f\|_h := \sqrt{(f, f)_h}.$$ 

**Definition.** For an antisymmetric function $\theta$, define its energy to be

$$\mathcal{E}(\theta) := \|\theta\|^2_r,$$

where $r$ is the collection of resistances.
Thus $E(i) = (i, i)_r = (i, dv)$. If $i$ is a unit current flow from $A$ to $Z$ with voltages $v_A$ and $v_Z$ constant on $A$ and on $Z$, respectively, then by Lemma 2.9 and Exercise 2.5,

$$E(i) = v_A - v_Z = R(A \leftrightarrow Z). \quad (2.6)$$

The inner product $(\cdot, \cdot)_r$ is important not only for its squared norm $E(\cdot)$. For example, we may express Kirchhoff’s laws as follows. Let $\chi^e := 1_{\{e\}} - 1_{\{-e\}}$ denote the unit flow along $e$ represented as an antisymmetric function in $\ell^2(E)$. Note that for every antisymmetric function $\theta$ and every $e$, we have

$$(\chi^e, \theta)_r = \theta(e)r(e),$$

so that

$$(\sum_{e^- = x} c(e)\chi^e, \theta)_r = d^*\theta(x). \quad (2.7)$$

Let $i$ be any current.

**Kirchhoff’s Node Law:** For every vertex $x$, we have

$$\left( \sum_{e^- = x} c(e)\chi^e, i \right)_r = 0$$

except if a battery is connected at $x$.

**Kirchhoff’s Cycle Law:** If $e_1, e_2, \ldots, e_n$ is an oriented cycle in $G$, then

$$\left( \sum_{k=1}^n \chi^{e_k}, i \right)_r = 0.$$
G is finite, the uniqueness principle implies that $F$ is constant on each component of $G$, whence $\theta = 0$, as desired.

Thus, Kirchhoff’s Cycle Law says that $i$, being orthogonal to $\diamond$, is in $\bigstar$. Furthermore, any $i \in \bigstar$ is a current by Exercise 2.2 (let $W := \{x; (d^*i)(x) \neq 0\}$). Now if $\theta$ is any flow with the same sources and sinks as $i$, more precisely, if $\theta$ is any antisymmetric function such that $d^*\theta = d^*i$, then $\theta - i$ is a sourceless flow, i.e., by (2.7), is orthogonal to $\bigstar$ and thus is an element of $\diamond$. Therefore, the expression

$$\theta = i + (\theta - i)$$

is the orthogonal decomposition of $\theta$ relative to $\ell^2(E) = \bigstar + \diamond$. This shows that the orthogonal projection $P_{\bigstar} : \ell^2(E) \to \bigstar$ plays a crucial role in network theory. In particular,

$$i = P_{\bigstar}\theta \quad (2.8)$$

and

$$\|\theta\|_r^2 = \|i\|_r^2 + \|\theta - i\|_r^2. \quad (2.9)$$

**Thomson’s Principle.** Let $G$ be a finite network and $A$ and $Z$ be two disjoint subsets of its vertices. Let $\theta$ be a flow from $A$ to $Z$ and $i$ be the current flow from $A$ to $Z$ with $d^*i = d^*\theta$. Then $E(\theta) > E(i)$ unless $\theta = i$.

*Proof.* The result is an immediate consequence of (2.9). \hfill □

Note that given $\theta$, the corresponding current $i$ such that $d^*i = d^*\theta$ is unique (and given by (2.8)).

Recall that $E_{1/2} \subset E$ is a set of edges containing exactly one of each pair $e, -e$. What is the matrix of $P_{\bigstar}$ in the orthogonal basis $\{\chi^e; e \in E_{1/2}\}$? We have

$$(P_{\bigstar}\chi^e, \chi^f)_r = (i^e, \chi^f)_r = i^e(f)r(f), \quad (2.10)$$

where $i^e$ is the unit current from $e^-$ to $e^+$. In other words, the matrix coefficient at $(f, e)$ is the voltage difference across $f$ when a unit current is imposed between the endpoints of $e$. This matrix is called the transfer impedance matrix. The related matrix with entries $Y(e, f) := i^e(f)$ is called, naturally enough, the transfer current matrix. This latter matrix will be extremely useful for our study of random spanning trees and forests in Chapters 8 and 9. Since $P_{\bigstar}$, being an orthogonal projection, is self-adjoint, the transfer impedance matrix is symmetric. Therefore

$$Y(e, f)r(f) = Y(f, e)r(e). \quad (2.11)$$
§4. Energy

This is called the **reciprocity law**.

Consider $P[a \rightarrow Z]$. How does this change when an edge is removed from $G$? when an edge is added? when the conductance of an edge is changed? These questions are not easy to answer probabilistically, but yield to the ideas we have developed. Since $P[a \rightarrow Z] = C(a \leftrightarrow Z)/\pi(a)$, if no edge incident to $a$ is affected, then we need analyze only the change in effective conductance.

**Exercise 2.12.**

Show that $P[a \rightarrow Z]$ can increase in some situations and decrease in others when an edge incident to $a$ is removed.

Effective conductance changes as follows. We use subscripts to indicate the edge conductances used.

**Rayleigh’s Monotonicity Law.** *Let $G$ be a finite graph and $A$ and $Z$ two disjoint subsets of its vertices. If $c$ and $c'$ are two assignments of conductances on $G$ with $c \leq c'$, then $C_c(A \leftrightarrow Z) \leq C_{c'}(A \leftrightarrow Z)$.*

**Proof.** By (2.6), we have $C(A \leftrightarrow Z) = 1/\mathcal{E}(i)$ for a unit current flow $i$ from $A$ to $Z$. Now

$$\mathcal{E}_c(i_c) \geq \mathcal{E}_{c'}(i_c) \geq \mathcal{E}_{c'}(i_{c'}) ,$$

where the first inequality follows from the definition of energy and the second from Thomson’s principle. Taking reciprocals gives the result. 

In particular, removing an edge decreases effective conductance, so if the edge is not incident to $a$, then its removal decreases $P[a \rightarrow Z]$. In addition, contracting an edge (called “shorting” in electrical network theory), i.e., identifying its two endpoints and removing the resulting loop, increases the effective conductance between any sets of vertices. This is intuitive from thinking of increasing to infinity the conductance on the edge to be contracted, so we will still refer to it as part of Rayleigh’s Monotonicity Law. To prove it rigorously, let $i$ be the unit current flow from $A$ to $Z$. If the graph $G$ with the edge $e$ contracted is denoted $G/\{e\}$, then the edge set of $G/\{e\}$ may be identified with $E(G) \setminus \{e\}$. If $e$ does not connect $A$ to $Z$, then the restriction $\theta$ of $i$ to the edges of $G/\{e\}$ is a unit flow from $A$ to $Z$, whence the effective resistance between $A$ and $Z$ in $G/\{e\}$ is at most $\mathcal{E}(\theta)$, which is at most $\mathcal{E}(i)$, which equals the effective resistance in $G$. 


DRAFT Version of 17 April 2005.
§2.5. Transience and Recurrence.

We have seen that effective conductance from any vertex is positive iff the random walk is transient. We will formulate this as an energy criterion.

If \( G = (V, E) \) is a denumerable network, let

\[
\ell^2(V) := \{ f : V \to \mathbb{R}; \sum_{x \in V} f(x)^2 < \infty \}
\]

with the inner product \( (f, g) := \sum_{x \in V} f(x)g(x) \). Define the Hilbert space

\[
\ell^2_-(E, r) := \{ \theta : E \to \mathbb{R}; \forall e \theta(-e) = -\theta(e) \text{ and } \sum_{e \in E} \theta(e)^2 r(e) < \infty \}
\]

with the inner product \( (\theta, \theta')_r := \sum_{e \in E \downarrow 2} \theta(e)\theta'(e)r(e) \) and \( \mathcal{E}(\theta) := (\theta, \theta)_r \). Define \( df(e) := f(e^-) - f(e^+) \) as before. If \( \sum_{e = x} |\theta(e)| < \infty \), then we also define \( (d^*\theta)(x) := \sum_{e = x} \theta(e) \).

If \( V \) is finite and \( \sum_e |\theta(e)| < \infty \), then the calculation of Exercise 2.11 shows that we still have \( (\theta, df) = (d^*\theta, f) \) for all \( f \). Likewise, under these hypotheses, we have Lemma 2.8 and Lemma 2.9 still holding. The remainder of Section 2.4 also holds because of the following consequence of the Cauchy-Schwarz inequality:

\[
\forall x \in V \sum_{e^- = x} |\theta(e)| \leq \sqrt{\sum_{e^- = x} \theta(e)^2/c(e) \cdot \sum_{e^- = x} c(e)} \leq \sqrt{\mathcal{E}(\theta)\pi(x)}. \tag{2.12}
\]

In particular, if \( \mathcal{E}(\theta) < \infty \), then \( d^*\theta \) is defined.

\[ \blacktriangleright \text{Exercise 2.13.} \]

Let \( G = (V, E) \) be denumerable and \( \theta_n \in \ell^2_-(E, r) \) be such that \( \mathcal{E}(\theta_n) \leq M < \infty \) and \( \forall e \in E \theta_n(e) \to \theta(e) \). Show that \( \theta \) is antisymmetric, \( \mathcal{E}(\theta) \leq \lim inf_n \mathcal{E}(\theta_n) \leq M \), and \( \forall x \in V \ d^*\theta_n(x) \to d^*\theta(x) \).

Call an antisymmetric function \( \theta \) on \( E \) a unit flow from \( a \in V \) to \( \infty \) if

\[
\forall x \in V \sum_{e^- = x} |\theta(e)| < \infty
\]

and \( (d^*\theta)(x) = 1\{x\}(x) \). Our main theorem is the following criterion for transience.

\[ \textbf{Theorem 2.10. (T. Lyons, 1983)} \]

Let \( G \) be a denumerable connected network. Random walk on \( G \) is transient iff there is a unit flow on \( G \) of finite energy from some (every) vertex to \( \infty \).

\[ \textbf{Proof.} \] Let \( G_n \) be finite subgraphs that exhaust \( G \). Let \( G_n^W \) be the graph obtained from \( G \) by identifying the vertices outside \( G_n \) to a single vertex, \( z_n \). In other words, identify
these vertices and throw away resulting loops (but keep multiple edges). Fix any vertex
\( a \in G \), which, without loss of generality, belongs to each \( G_n \). We have, by definition,
\( R(a \leftrightarrow \infty) = \lim R(a \leftrightarrow z_n) \). Let \( i_n \) be the unit current flow in \( G_n^W \) from \( a \) to \( z_n \). Then
\[ E(i_n) = R(a \leftrightarrow z_n), \]
so \( R(a \leftrightarrow \infty) < \infty \iff \lim E(i_n) < \infty \).

Note that each edge of \( G_n^W \) comes from an edge in \( G \) and may be identified with it, even though one endpoint may be different.

If \( \theta \) is a unit flow on \( G \) from \( a \) to \( \infty \) and of finite energy, then the restriction \( \theta|G_n^W \) of \( \theta \) to \( G_n^W \) is a unit flow from \( a \) to \( z_n \), whence Thomson’s principle gives \( E(i_n) \leq E(\theta|G_n^W) \leq E(\theta) < \infty \). In particular, \( \lim E(i_n) < \infty \) and so the random walk is transient.

Conversely, if \( E(i_n) \leq M < \infty \), then (by a diagonal argument) we may find a subsequence \( i_{n_k} \) converging pointwise (i.e., edgewise) to some function \( \theta \). Exercise 2.13 ensures that \( \theta \) is a unit flow of finite energy from \( a \) to \( \infty \).

This allows us to carry over the remainder of the electrical apparatus to infinite networks:

**Proposition 2.11.** Let \( G \) be a transient connected network and \( G_n \) be finite subnetworks
containing a vertex \( a \) that exhaust \( G \). Identify the vertices outside \( G_n \) to \( z_n \), forming \( G_n^W \).
Let \( i_n \) be the unit current flow in \( G_n^W \) from \( a \) to \( z_n \). Then \( \langle i_n \rangle \) has a pointwise limit \( i \) on \( G \), which is the unique unit flow on \( G \) from \( a \) to \( \infty \) of minimum energy. Let \( v_n \) be the
voltages on \( G_n^W \) corresponding to \( i_n \) and with \( v_n(z_n) := 0 \). Then \( v := \lim v_n \) exists on \( G \)
and has the following properties:

\[
\begin{align*}
\text{dv} &= i, \\
v(a) &= E(i) = R(a \leftrightarrow \infty), \\
\forall x \quad v(x)/v(a) &= P_x[\tau_a < \infty].
\end{align*}
\]

Start a random walk at \( a \). For all vertices \( x \), the expected number of visits to \( x \) is \( G(a, x) = \pi(x)\nu(x) \).
For all edges \( e \), the expected signed number of crossings of \( e \) is \( i(e) \).

**Proof.** We first establish uniqueness of a unit flow (from \( a \) to \( \infty \)) with minimum energy.
Note that \( \forall \theta, \theta' \)
\[
\frac{E(\theta) + E(\theta')}{2} = E\left(\frac{\theta + \theta'}{2}\right) + E\left(\frac{\theta - \theta'}{2}\right).
\]
Therefore, if \( \theta \) and \( \theta' \) both have minimum energy, so does \( (\theta + \theta')/2 \) and hence \( E((\theta - \theta')/2) = 0 \), which gives \( \theta = \theta' \).

Now let \( \theta \) be a unit flow of finite energy. As in the proof of the theorem, \( E(i_n) \leq E(\theta) \),
whence every limit point \( i \) of \( \{i_n\} \) has energy \( \leq E(\theta) \). Thus, \( i \) is the unique flow of minimum
energy and so \( i_n \to i \).
On the other hand, by Exercise 2.13, \( \mathcal{E}(i) \leq \lim \inf \mathcal{E}(i_n) \). Since \( \mathcal{E}(i) \leq \mathcal{E}(i_n) \), we have \( \mathcal{E}(i) = \lim \mathcal{E}(i_n) = \lim v_n(a) = v(a) \). This also equals \( \lim \mathcal{R}(a \leftrightarrow z_n) = \mathcal{R}(a \leftrightarrow \infty) \).

Since the events \( \{\tau_a < \tau_{G \setminus G_n}\} \) are increasing in \( n \) with union \( \{\tau_a < \infty\} \), we have (with superscript indicating on which network the random walk takes place)

\[
\lim_n P_{x_n}^{G_n} [\tau_a < \tau_{z_n}] = \lim_n P_{x}^{G} [\tau_a < \tau_{G \setminus G_n}] = P_{x}^{G} [\tau_a < \infty],
\]

whence \( \lim v_n(x)/v_n(a) = v(x)/v(a) \) exists and equals \( P_{x}^{G} [\tau_a < \infty] \). Since \( dv_n = i_n r \), taking limits gives \( dv = ir \).

Let \( Y_n(x) \) be the number of visits to \( x \) before hitting \( G \setminus G_n \) and \( Y(x) \) be the total number of visits to \( x \). Then \( Y_n(x) \) increases to \( Y(x) \), whence the Monotone Convergence Theorem implies that \( G(a, x) = \mathbb{E}[Y(x)] = \lim_{n \to \infty} \mathbb{E}[Y_n(x)] = \lim_{n \to \infty} \pi(x)v_n(x) = \pi(x)v(x) \). Finally, the proof of Proposition 2.2 now applies as written for the last claim of the proposition. \( \blacksquare \)

Thus, we may call \( i \) the unit current flow and \( v \) the voltage on \( G \). We may regard \( G \) as grounded (i.e., has 0 voltage) at infinity.

By Theorem 2.10 and Rayleigh’s monotonicity law, the type of a random walk, i.e., whether it is transient or recurrent, does not change when the conductances are changed by bounded factors. An extensive generalization of this is given in Theorem 2.16.

How do we determine whether there is a flow from \( a \) to \( \infty \) of finite energy? There is no recipe, but there is a very useful necessary condition due to Nash-Williams (1959) and some useful techniques. A set \( \Pi \) of edges separates \( a \) and \( \infty \) if every infinite simple path from \( a \) must include an edge in \( \Pi \); we also call \( \Pi \) a cutset.

**The Nash-Williams Criterion.** If \( \langle \Pi_n \rangle \) is a sequence of pairwise disjoint cutsets in a network \( G \) that separate \( a \) from \( \infty \), then

\[
\mathcal{R}(a \leftrightarrow \infty) \geq \sum_n \left( \sum_{e \in \Pi_n} c(e) \right)^{-1}. \tag{2.14}
\]

In particular, if the right-hand side is infinite, then \( G \) is recurrent.

**Proof.** By Proposition 2.11, it suffices to show that every unit flow \( \theta \) from \( a \) to \( \infty \) has energy at least the right-hand side. The Cauchy-Schwarz inequality gives

\[
\sum_{e \in \Pi_n} \theta(e)^2 r(e) \sum_{e \in \Pi_n} c(e) \geq \left( \sum_{e \in \Pi_n} |\theta(e)| \right)^2 \geq 1.
\]
Hence
\[ E(\theta) \geq \sum_n \sum_{e \in \Pi_n} \theta(e)^2 r(e) \geq \sum_n \left( \sum_{e \in \Pi_n} c(e) \right)^{-1}. \]

**Remark 2.12.** If the cutsets can be ordered so that \( \Pi_1 \) separates \( a \) from \( \Pi_2 \) and for \( n > 1 \), \( \Pi_n \) separates \( \Pi_{n-1} \) from \( \Pi_{n+1} \), then the sum appearing in the statement of this criterion has a natural interpretation: Short together (i.e., join by edges of infinite conductance, or, in other words, identify) all the vertices between \( \Pi_n \) and \( \Pi_{n+1} \) into one vertex \( U_n \). Short all the vertices that \( \Pi_1 \) separates from \( \infty \) into one vertex \( U_0 \). Then only parallel edges of \( \Pi_n \) join \( U_{n-1} \) to \( U_n \). Replace these edges by a single edge of resistance \( \left( \sum_{e \in \Pi_n} c(e) \right)^{-1} \). This new network is a series network with effective resistance from \( U_0 \) to \( \infty \) equal to the right-hand side of (2.14). Thus, Rayleigh’s monotonicity law shows that the effective resistance from \( a \) to \( \infty \) in \( G \) is at least the right-hand side of (2.14).

The criterion of Nash-Williams is very useful for proving recurrence. In order to prove transience, there is a very useful way to define flows via random paths. Suppose that \( P \) is a probability measure on paths \( \langle e_n \rangle \) from \( a \) to \( z \) on a finite graph or from \( a \) to \( \infty \) on an infinite graph. Define
\[ \theta(e) := \sum_{n \geq 0} (P[e_n = e] - P[e_n = -e]) \]  
(2.15)
provided
\[ \sum_{n \geq 0} (P[e_n = e] + P[e_n = -e]) < \infty. \]

For example, the summability condition holds when the paths are edge-simple. Each path \( \langle e_n \rangle \) determines a unit flow \( \psi \) from \( a \) to \( z \) (or to \( \infty \)) by sending 1 along each edge in the path:
\[ \psi := \sum_{n \geq 0} \chi^{e_n}. \]

Since \( \theta \) is an expectation of unit flows, we get that \( \theta \) is a unit flow itself. We saw in Propositions 2.2 and 2.11 that this is precisely how network random walks and unit electric current are related (where the walk \( \langle X_n \rangle \) gives rise to the path \( \langle e_n \rangle \) with \( e_n := \langle X_n, X_{n+1} \rangle \)). However, there are other useful pairs of random paths and their expected flows as well.

We now illustrate the preceding techniques. First, we prove Pólya’s famous theorem concerning random walk on the integer lattices. This proof is a slight modification of T. Lyons’s 1983 proof.

**Pólya’s Theorem.** Simple random walk on the nearest-neighbor graph of \( \mathbb{Z}^d \) is recurrent for \( d = 1, 2 \) and transient for all \( d \geq 3 \).
Proof. For $d = 1, 2$, we can use the Nash-Williams criterion with cutsets 

$$\Pi_n := \{ e : d(o, e^-) = n, \ |d(o, e^-) - d(o, e^+)| = 1 \},$$

where $o$ is the origin and $d(\cdot, \cdot)$ is the graph distance.

For the other cases, by Rayleigh’s Monotonicity Law, it suffices to do $d = 3$. Let $L$ be a random uniformly distributed ray from the origin $o$ of $\mathbb{R}^3$ to $\infty$. Let $\mathcal{P}(L)$ be a simple path in $\mathbb{Z}^3$ from $o$ to $\infty$ that stays within distance 4 of $L$; choose $\mathcal{P}(L)$ measurably, such as the (almost surely unique) closest path to $L$. Define the flow $\theta$ from the law of $\mathcal{P}(L)$ via (2.15). Then $\theta$ is a unit flow from $o$ to $\infty$; we claim it has finite energy. There is some constant $A$ such that if $e$ is an edge whose midpoint is at euclidean distance $R$ from $o$, then $\mathbb{P}[e \in \mathcal{P}(L)] \leq A/R^2$. Since all edge centers are separated from each other by euclidean distance at least 1, there is also a constant $B$ such that there are at most $Bn^2$ edge centers whose distance from the origin is between $n$ and $n + 1$. It follows that the energy of $\theta$ is at most $\sum_n A^2 Bn^2 n^{-4}$, which is finite. Now transience follows from Theorem 2.10. ▶

Remark 2.13. The continuous case, i.e., Brownian motion in $\mathbb{R}^3$, is easier to handle (after establishing a similar relationship to an electrical framework) because of the spherical symmetry; see Section 2.10, the notes to this chapter. Here, we are approximating this continuous case in our solution. One can in fact use the transience of the continuous case to deduce that of the discrete case (or vice versa); see Theorem 2.32 in the notes.

Since simple random walk on $\mathbb{Z}^2$ is recurrent, the effective resistance from the origin to distance $n$ tends to infinity—but how quickly? Our techniques are good enough to answer this within a constant factor. Note first that the proof of (2.14) shows that if $a$ and $z$ are separated by pairwise disjoint cutsets $\Pi_1, \ldots, \Pi_n$, then

$$\mathcal{R}(a \leftrightarrow z) \geq \sum_{k=1}^n \left( \sum_{e \in \Pi_k} c(e) \right)^{-1}. \tag{2.16}$$

Proposition 2.14. There are positive constants $C_1, C_2$ such that if one identifies to a single vertex $z_n$ all vertices of $\mathbb{Z}^2$ that are at distance more than $n$ from $o$, then

$$C_1 \log n \leq \mathcal{R}(o \leftrightarrow z_n) \leq C_2 \log n.$$ 

Proof. The lower bound is an immediate consequence of (2.16) applied to the cutsets $\Pi_k$ used in our proof of Pólya’s theorem. The upper bound follows from the estimate of the energy of the unit flow analogous to that used for the transience of $\mathbb{Z}^3$. That is, $\theta(e)$ is defined via (2.15) from a uniform ray emanating from the origin. Then $\theta$ defines a unit flow from $o$ to $z_n$ and its energy is bounded by $C_2 \log n$. ▶
We can extend Proposition 2.14 as follows.

**Proposition 2.15.** For $d \geq 2$, there is a positive constant $C_d$ such that if $G_n$ is the subnetwork of $\mathbb{Z}^d$ induced on the vertices in a box of side length $n$, then for any pair of vertices $x, y$ in $G_n$ at mutual distance $k$,

$$\mathcal{R}(x \leftrightarrow y; G_n) \in \begin{cases} (C_d^{-1} \log k, C_d \log k) & \text{if } d = 2, \\ (C_d^{-1}, C_d) & \text{if } d \geq 3. \end{cases}$$

**Proof.** The lower bounds follow from (2.16). For the upper bounds, we give the details for $d = 2$ only. There is a straight-line segment $L$ of length $k$ inside the portion of $\mathbb{R}^2$ that corresponds to $G_n$ such that $L$ meets the straight line $M$ joining $x$ and $y$ at the midpoint of $M$ in a right angle. Let $Q$ be a random uniform point on $L$. Write $L(Q)$ for the union of the straight-line segments between $x, y$ and $M$. Let $P(Q)$ be a path in $G_n$ from $x$ to $y$ that is closest to $L(Q)$. Use the law $P$ of $P(Q)$ to define the unit flow $\theta$ as in (2.15). Then $E(\theta) \leq C_2 \log k$ for some $C_2$, as in the proof of Proposition 2.14. \hfill ▵

Since the harmonic series, which arises in the recurrence of $\mathbb{Z}^2$, just barely diverges, it seems that the change from recurrence to transience occurs “just after” dimension 2, rather than somewhere else in $[2, 3]$. One way to make sense of this is to ask about the type of spaces intermediate between $\mathbb{Z}^2$ and $\mathbb{Z}^3$. For example, consider the wedge

$$W_f := \{(x, y, z); |z| \leq f(|x|)\},$$

where $f : \mathbb{N} \to \mathbb{N}$ is an increasing function. The number of edges that leave $W_f \cap \{(x, y, z); |x| \vee |y| \leq n\}$ is of the order $n(f(n) + 1)$, so that according to the Nash-Williams criterion,

$$\sum_{n \geq 1} \frac{1}{n(f(n) + 1)} = \infty \quad (2.17)$$

is sufficient for recurrence.

*Exercise 2.14.*

Show that (2.17) is also necessary for recurrence if $f(n + 1) \leq f(n) + 1$ for all $n$.

The most direct proof of Pólya’s theorem goes by calculation of the Green function and is not hard; see Exercise 2.85. However, that calculation depends on the precise structure of the graph. The proof here begins to show that the type doesn’t change when fairly drastic changes are made in the lattice graph. Suppose, for example, that diagonal edges are added to the square lattice. Then clearly we can still use Nash-Williams’ criterion.
to show recurrence. Of course, a similar addition of edges in higher dimensions preserves transience simply by Rayleigh’s Monotonicity Law. But suppose that in $\mathbb{Z}^3$, we remove each edge $[(x, y, z), (x, y, z + 1)]$ with $x + y$ odd. Is the resulting graph still transient? If so, by how much has the effective resistance to infinity changed?

Notice that graph distances haven’t changed much after these edges are removed. In a general network, if we think of the resistances $r$ as the lengths of edges, then we are led to the following definition.

Given two networks $G$ and $G'$ with resistances $r$ and $r'$, we say that a map $\phi$ from the vertices of $G$ to the vertices of $G'$ is a **rough embedding** if there are constants $\alpha, \beta < \infty$ and a map $\Phi$ defined on the edges of $G$ such that

(i) for every edge $\langle x, y \rangle \in G$, $\Phi(\langle x, y \rangle)$ is a non-empty simple oriented path of edges in $G'$ joining $\phi(x)$ and $\phi(y)$ with

$$\sum_{e' \in \Phi(\langle x, y \rangle)} r'(e') \leq \alpha r(x, y)$$

and $\Phi(\langle y, x \rangle)$ is the reverse of $\Phi(\langle x, y \rangle)$;

(ii) for every edge $e' \in G'$, there are no more than $\beta$ edges in $G$ whose image under $\Phi$ contains $e'$.

If we need to refer to the constants, we call the map $(\alpha, \beta)$-**rough**. We call two networks **roughly equivalent** if there are rough embeddings in both directions. For example, every two euclidean lattices of the same dimension are roughly equivalent.

**Theorem 2.16.** (Kanai, 1986) If $G$ and $G'$ are roughly equivalent connected networks, then $G$ is transient iff $G'$ is transient. In fact, if there is a rough embedding from $G$ to $G'$ and $G$ is transient, then $G'$ is transient.

**Proof.** Suppose that $G$ is transient and $\phi$ is an $(\alpha, \beta)$-rough embedding from $G$ to $G'$. Let $\theta$ be a unit flow on $G$ of finite energy from $a$ to infinity. We will use $\Phi$ to carry the flow $\theta$ to a unit flow $\theta'$ on $G'$ that will have finite energy. Namely, define

$$\theta'(e') := \sum_{e' \in \Phi(e)} \theta(e).$$

(The sum goes over all edges, not merely those in $E_{1/2}$.) It is easy to see that $\theta$ is antisymmetric and $d^*\theta'(y) = \sum_{x \in \phi^{-1}(\{y\})} d^*\theta(x)$ for all $y \in G'$. Thus, $\theta'$ is a unit flow from $\phi(a)$ to infinity.

Now

$$\theta'(e')^2 \leq \beta \sum_{e' \in \Phi(e)} \theta(e)^2$$
by the Cauchy-Schwarz inequality and the condition (ii). Therefore,
\[
\sum_{e' \in E'} \theta'(e') r'(e') \leq \beta \sum_{e' \in E'} \sum_{e' \in \Phi(e)} \theta(e) r'(e') = \beta \sum_{e \in E} \sum_{e' \in \Phi(e)} \theta(e) r'(e') \\
\leq \alpha \beta \sum_{e \in E} \theta(e) r(e) < \infty.
\]

\[\Box\]

\[\textbf{Exercise 2.15.}\]

Show that the removal of edges in \(\mathbb{Z}^3\) as above gives a transient graph with effective resistance to infinity at most 6 times what it was before removal.

A closely related notion is that of rough isometry, also called quasi-isometry. Given two graphs \(G = (V, E)\) and \(G' = (V', E')\), call a function \(\phi : V \to V'\) a \textbf{rough isometry} if there are positive constants \(\alpha\) and \(\beta\) such that for all \(x, y \in V\),
\[
\alpha^{-1} d(x, y) - \beta \leq d'(\phi(x), \phi(y)) \leq \alpha d(x, y) + \beta \tag{2.18}
\]
and such that every vertex in \(G'\) is within distance \(\beta\) of the image of \(V\). Here, \(d\) and \(d'\) denote the usual graph distances on \(G\) and \(G'\). In fact, the same definition applies to metric spaces, with “vertex” replaced by “point”.

\[\textbf{Exercise 2.16.}\]

Show that being roughly isometric is an equivalence relation.

\[\textbf{Proposition 2.17.}\] Let \(G\) and \(G'\) be two infinite roughly isometric networks with conductances \(c\) and \(c'\). If \(c, c', c^{-1}, c'^{-1}\) are all bounded and the degrees in \(G\) and \(G'\) are all bounded, then \(G\) is roughly equivalent to \(G'\).

\[\textbf{Exercise 2.17.}\]

Prove Proposition 2.17.

We now consider graphs that are roughly isometric to hyperbolic spaces. Let \(\mathbb{H}^d\) denote the standard hyperbolic space of dimension \(d \geq 2\); it has scalar curvature \(-1\) everywhere. See Figure 2.1 for one such graph, drawn by a program created by Don Hatch. This drawing uses the Poincaré disc model of \(\mathbb{H}^2\), in which the unit disc \(\{z \in \mathbb{C} ; |z| < 1\}\) is given the arc-length metric \(2|dz|/(1 - |z|^2)\). The corresponding ball model of \(\mathbb{H}^d\) uses the unit ball \(\{x \in \mathbb{R}^d ; |x| < 1\}\) with the arc-length metric \(2|dx|/(1 - |x|^2)\). For each point \(a\) in the ball, there is a hyperbolic isometry that takes \(a\) to the origin, namely,
\[
x \mapsto a^* + \frac{|a^*|^2 - 1}{|x - a^*|^2} (x - a^*),
\]
where $a^* := a/|a|^2$; see, e.g., Matsuzaki and Taniguchi (1998) for the calculation. It follows easily from this that for each point $o \in \mathbb{H}^d$, the sphere of hyperbolic radius $r$ about $o$ has surface area asymptotic to $\alpha e^{r(d-1)}$ for some positive constant $\alpha$ depending on $d$. Indeed, if $|x| = R$, then the distance between the origin and $x$ is

$$r = \int_0^R \frac{2ds}{1-s^2} = \log \frac{1+R}{1-R},$$

so that

$$R = \frac{e^r - 1}{e^r + 1}.$$ 

The surface area of the sphere centered at the origin is therefore

$$\int_{|x|=R} \frac{2^{d-1}dS}{(1-|x|^2)^{d-1}},$$

where $dS$ is the element of surface area in $\mathbb{R}^d$. Integrating gives the result

$$C \left( \frac{R}{1-R^2} \right)^{d-1} = C(e^r - e^{-r})^{d-1}$$

for some constant $C$. Therefore there is a positive constant $A$ such that the following hold for any fixed point $o \in \mathbb{H}^d$:

1. the volume of the shell of points whose distance from $o$ is between $r$ and $r + 1$ is at most $A e^{r(d-1)}$;
2. the solid angle subtended at $o$ by a spherical cap of area $\delta$ on the sphere centered at $o$ of radius $r$ is at most $A \delta e^{-r(d-1)}$.

For more background on hyperbolic space, see, e.g., Ratcliffe (1994) or Benedetti and Petronio (1992). Graphs that are roughly isometric to $\mathbb{H}^d$ often arise as Cayley graphs of groups (see Section 2.9) or, more generally, as nets. A graph $G$ is called an $\epsilon$-net of a metric space $M$ if the vertices of $G$ form a maximal $\epsilon$-separated subset of $M$ and edges join distinct vertices iff their distance in $M$ is at most $3\epsilon$.

**Theorem 2.18.** If $G$ is roughly isometric to a hyperbolic space $\mathbb{H}^d$, then simple random walk on $G$ is transient.

**Proof.** By Theorem 2.16, given $d \geq 2$, it suffices to show transience for one such $G$. Let $G$ be a 1-net of $\mathbb{H}^d$. Let $L$ be a random uniformly distributed geodesic ray from some point $o \in G$ to $\infty$. Let $\mathcal{P}(L)$ be a simple path in $G$ from $o$ to $\infty$ that stays within distance 1 of $L$; choose $\mathcal{P}(L)$ measurably. (By choice of $G$, for all $p \in L$, there is a vertex $x \in G$ within distance 1 of $p$.) Define the flow $\theta$ from the law of $\mathcal{P}(L)$ via (2.15). Then $\theta$ is a unit flow from $o$ to $\infty$; we
claim it has finite energy. There is some constant $C$ such that if $e$ is an edge whose midpoint is at hyperbolic distance $r$ from $o$, then $P[e \in P(L)] \leq Ce^{-r(d-1)}$. Since all edge centers are separated from each other by hyperbolic distance at least 1, there is also a constant $D$ such that there are at most $De^{n(d-1)}$ edge centers whose distance from the origin is between $n$ and $n+1$. It follows that the energy of $\theta$ is at most $\sum_n C^2De^{-2n(d-1)}e^{n(d-1)}$, which is finite. Now transience follows from Theorem 2.10.

\section*{§2.6. Flows, Cutsets, and Random Paths.}

Notice that if there is a flow from $a$ to $\infty$ of finite energy on some network and if $i$ is the unit current flow, then $|i| = |c \cdot dv| \leq v(a)c$. In particular, there is a non-0 flow bounded on each edge by $c$ (viz., $i/v(a)$). The existence of flows that are bounded by some given numbers on the edges is an interesting and valuable property in itself. To determine whether there is a non-0 flow bounded by $c$, we turn to the Max-Flow Min-Cut Theorem of Ford and Fulkerson (1962). For finite networks, the theorem reads as follows. We call a set $\Pi$ of edges a \textbf{cutset} separating $A$ and $Z$ if every path joining a vertex in $A$ to a vertex