ISOMETRIES AND ISOMORPHISMS
IN QUASI-BANACH ALGEBRAS

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Abstract. In this paper, we prove the Hyers-Ulam-Rassias stability of isometries and of homomorphisms for additive functional equations in quasi-Banach algebras. This is applied to investigate isomorphisms between quasi-Banach algebras.

1. Introduction and preliminaries

Stability is investigated when one concerns whether a small error of parameters causes a large deviation of the solution. Generally speaking, given a function which satisfies a functional equation approximately called an approximate solution, we ask: Is there a solution of this equation which is close to the approximate solution in some accuracy? An earlier work was done by Hyers [11] in order to answer Ulam’s question ([20]) on approximately additive mappings. Later there have been given lots of results on stability in the Hyers-Ulam sense or some generalized sense (see books and papers [1, 3, 8, 9, 12, 17, 18] and references therein).

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G. Z. Eskandani [7] established the general solution and investigated the Hyers-Ulam-Rassias stability of the following functional equation

\[
\sum_{i=1}^{m} f \left( mx_i + \sum_{j=1, j \neq i}^{m} x_j \right) + f \left( \sum_{i=1}^{m} x_i \right) = 2f \left( \sum_{i=1}^{m} mx_i \right)
\]

in quasi-Banach spaces, where \( m \in \mathbb{N} \) and \( m \geq 2 \). The stability of isometries in norms spaces and Banach spaces was investigated in several papers [4, 6, 10, 13]. However, C. Park and Th. M. Rassias [15] proved the Hyers-Ulam stability of isometric additive functional equations in quasi-Banach spaces. C. Park [16] studied the Hyers-Ulam stability of homomorphisms in quasi-Banach algebras. Recently, M. S. Moslehian and Gh. Sadeghi [14] have proved the Hyers-Ulam-Rassias stability of linear mappings in quasi-Banach modules associated to the Cauchy functional equation and a generalized Jensen functional equation.

The main purpose of this paper is to study the Hyers-Ulam-Rassias stability of equation (1.1). More precisely, we prove the Hyers-Ulam-Rassias stability of isometric additive functional equations (1.1) in quasi-Banach algebras. Furthermore, we investigate the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to additive functional equations (1.1). This is applied to investigate isomorphisms between quasi-Banach algebras.

We now give some basic facts concerning quasi-Banach spaces and some preliminary results.

**Definition 1.1** (cf. [5, 19]). Let \( X \) be a real linear space. A quasi-norm is a real-valued function on \( X \) satisfying the following:

1. \( \| x \| \geq 0 \) for all \( x \in X \) and \( \| x \| = 0 \) if and only if \( x = 0 \).
2. \( \| \lambda x \| = |\lambda| \cdot \| x \| \) for all \( \lambda \in \mathbb{R} \) and for all \( x \in X \).
3. There is a constant \( K \geq 1 \) such that \( \| x + y \| \leq K(\| x \| + \| y \|) \) for all \( x, y \in X \).
The pair \((X, \| \cdot \|)\) is called a \textit{quasi-normed space} if \(\| \cdot \|\) is a \textit{quasi-norm} on \(X\). The smallest possible \(K\) is called the \textit{modulus of concavity} of \(\| \cdot \|\). A \textit{quasi-Banach space} is a complete \textit{quasi-normed space}.

A \textit{quasi-norm} \(\| \cdot \|\) is called a \textit{p-norm} \((0 < p \leq 1)\) if
\[
\|x + y\|^p \leq \|x\|^p + \|y\|^p
\]
for all \(x, y \in X\). In this case, a \textit{quasi-Banach space} is called a \textit{p-Banach space}.

Given a \(p\)-norm, the formula \(d(x, y) := \|x - y\|^p\) gives us a translation invariant metric on \(X\). By the Aoki-Rolewicz theorem \([19]\) (see also \([5]\)), each quasi-norm is equivalent to some \(p\)-norm. Since it is much easier to work with \(p\)-norms than quasi-norms, henceforth we restrict our attention mainly to \(p\)-norms.

**Definition 1.2** (cf. \([2]\)). Let \((X, \| \cdot \|)\) be a quasi-normed space. The quasi-normed space \((X, \| \cdot \|)\) is called a \textit{quasi-normed algebra} if \(X\) is an algebra and there is a constant \(C > 0\) such that \(\|xy\| \leq C\|x\|\|y\|\) for all \(x, y \in X\).

A \textit{quasi-Banach algebra} is a complete \textit{quasi-normed algebra}. If the quasi-norm \(\| \cdot \|\) is a \(p\)-norm, then the \textit{quasi-Banach algebra} is called a \textit{p-Banach algebra}.

**Definition 1.3** (cf. \([15]\)). Let \(X\) and \(Y\) be quasi-Banach algebras with norms \(\| \cdot \|_X\) and \(\| \cdot \|_Y\), respectively. An additive mapping \(A: X \to Y\) is called an isometric additive mapping if the additive mapping \(A: X \to Y\) satisfies
\[
\|A(x) - A(y)\|_Y = \|x - y\|_X
\]
for all \(x, y \in X\).
2. Stability of isometric additive mappings in quasi-Banach algebras

Throughout this section and Section 3, assume that $X$ is a quasi-normed algebra with quasi-norm $\| \cdot \|_X$ and that $Y$ is a $p$-Banach algebra with $p$-norm $\| \cdot \|_Y$. Let $K$ be the modulus of concavity of $\| \cdot \|_Y$. For convenience, we use the following abbreviation for a given mapping $f: X \to Y$:

$$Df(x_1, \ldots, x_m) = \sum_{i=1}^{m} f \left( mx_i + \sum_{j=1, j \neq i}^{m} x_j \right) + f \left( \sum_{i=1}^{m} x_i \right) - 2f \left( \sum_{i=1}^{m} mx_i \right)$$

for all $x_j \in X$ ($1 \leq j \leq m$). We prove the Hyers-Ulam-Rassias stability of the isometric additive functional equation (1.1) in quasi-Banach algebras.

**Theorem 2.1.** Let $\varphi: X^m \to [0, \infty)$ be a mapping such that

(2.1) \[ \lim_{n \to \infty} \frac{1}{m^n} \varphi(m^n x_1, \ldots, m^n x_m) = 0 \]

(2.2) \[ \tilde{\varphi}(x) := \sum_{i=0}^{\infty} \frac{1}{m^{ip}} (\varphi(m^i x, 0, \ldots, 0))^p < \infty \]

for all $x, x_j \in X$ ($1 \leq j \leq m$). Suppose that a mapping $f: X \to Y$ satisfies

(2.3) \[ \|Df(x_1, \ldots, x_m)\|_Y \leq \varphi(x_1, \ldots, x_m) \]

(2.4) \[ |\|f(x)\|_Y - \|x\|_X| \leq \varphi(x, \ldots, x) \]

$m$-times
for all \( x, x_j \in X \ (1 \leq j \leq m) \). Then there exists a unique isometric additive mapping \( A: X \to Y \) such that

\[
\| f(x) - A(x) \|_Y \leq \frac{1}{m} [\tilde{\varphi}(x)]^{\frac{1}{p}}
\]

(2.5)

for all \( x \in X \).

Proof. By the Eskandani’s theorem [7, Theorem 2.2], it follows from (2.1), (2.2) and (2.3) that there exists a unique additive mapping \( A: X \to Y \) satisfying (2.5). The additive mapping \( A: X \to Y \) is given by

\[
A(x) := \lim_{n \to \infty} \frac{1}{m^n} f(m^n x)
\]

(2.6)

for all \( x \in X \).

It follows from (2.4) that

\[
\left| \| \frac{1}{m^n} f(m^n x) \|_Y - \| x \|_X \right| \leq \frac{1}{m^n} \left| \| f(m^n x) \|_Y - \| m^n x \|_X \right|
\]

\[
\leq \frac{1}{m^n} \varphi(m^n x, \ldots, m^n x)_{m \text{-times}}
\]

which tends to zero as \( n \to \infty \) for all \( x \in X \). So

\[
\| A(x) \|_Y = \lim_{n \to \infty} \| \frac{1}{m^n} f(m^n x) \|_Y = \| x \|_X
\]

for all \( x \in X \). Since \( A: X \to Y \) is additive,

\[
\| A(x) - A(y) \|_Y = \| A(x - y) \|_Y = \| x - y \|_X
\]

for all \( x \in X \). So the mapping \( A: X \to Y \) is an isometry. Thus the mapping \( A: X \to Y \) is a unique isometric additive mapping satisfying (2.5). This completes the proof of the theorem. \( \Box \)
Theorem 2.2. Let $\phi : X^m \to [0, \infty)$ be a mapping such that

\begin{align}
(2.7) \quad \lim_{n \to \infty} m^n \phi(\frac{x_1}{m^n}, \cdots, \frac{x_m}{m^n}) &= 0 \\
(2.8) \quad \tilde{\phi}(x) := \sum_{i=1}^{\infty} m^{ip}(\phi(\frac{x}{m^i}, 0, \cdots, 0))^p < \infty
\end{align}

for all $x, x_j \in X$ ($1 \leq j \leq m$). Suppose that a mapping $f : X \to Y$ satisfies

\begin{align}
(2.9) \quad \|Df(x_1, \cdots, x_m)\|_Y &\leq \phi(x_1, \cdots, x_m) \\
(2.10) \quad |\|f(x)\|_Y - \|x\|_X| &\leq \phi(x, \cdots, x)_{m\text{-times}}
\end{align}

for all $x, x_j \in X$ ($1 \leq j \leq m$). Then there exists a unique isometric additive mapping $A : X \to Y$ such that

\begin{align}
(2.11) \quad \|f(x) - A(x)\|_Y &\leq \frac{1}{m}[\tilde{\phi}(x)]^{\frac{1}{p}}
\end{align}

for all $x \in X$.

Proof. By the Eskandani’s theorem [7, Theorem 2.3], it follows from (2.7), (2.8) and (2.9) that there exists a unique additive mapping $A : X \to Y$ satisfying (2.11). The additive mapping $A : X \to Y$ is given by

\begin{align}
(2.12) \quad A(x) := \lim_{n \to \infty} m^n f(\frac{x}{m^n})
\end{align}

for all $x \in X$. 
By (2.10), we have
\[
|\|m^n f(\frac{x}{m^n})\|_Y - \|x\|_X| \leq m^n |\|f(\frac{x}{m^n})\|_Y - \|\frac{x}{m^n}\|_X|
\leq m^n \varphi(\frac{x}{m^n}, \cdots, \frac{x}{m^n})^{m\text{-times}}
\]
which tends to zero as \(n \to \infty\) for all \(x \in X\). By (2.12), we obtain
\[
\|A(x)\|_Y = \lim_{n \to \infty} \|m^n f(\frac{x}{m^n})\|_Y = \|x\|_X
\]
for all \(x \in X\). Hence
\[
\|A(x) - A(y)\|_Y = \|A(x - y)\|_Y = \|x - y\|_X
\]
for all \(x \in X\). So the additive mapping \(A: X \to Y\) is an isometry. This completes the proof of the theorem. \(\square\)

**Corollary 2.1.** Let \(\theta, r_j \ (1 \leq j \leq m)\) be non-negative real numbers such that \(r_j > 1\) or \(0 < r_j < 1\). Suppose that a mapping \(f: X \to Y\) satisfies
\[
\|Df(x_1, \cdots, x_m)\|_Y \leq \theta \sum_{i=1}^{m} \|x_i\|_X^{r_i}
\]
\[
|\|f(x)\|_Y - \|x\|_X| \leq \theta \sum_{i=1}^{m} \|x\|_X^{r_i}
\]
for all \(x, x_j \in X \ (1 \leq j \leq m)\). Then there exists a unique isometric additive mapping \(A: X \to Y\) such that
\[
\|f(x) - A(x)\|_Y \leq \frac{\theta}{|m^p - m^{pr_1}|^{\frac{1}{p}}} \|x\|_X^{r_1}
\]
for all $x \in X$.

Proof. The result follows from the proofs of Theorems 2.1 and 2.2. □

3. Stability of homomorphisms in quasi-Banach algebras

We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras, associated to the additive functional equation (1.1).

Theorem 3.1. Suppose that a mapping $f : X \to Y$ satisfies

\begin{align*}
\|Df(x_1, \cdots, x_m)\|_Y &\leq \varphi(x_1, \cdots, x_m) \\
\|f(xy) - f(x)f(y)\|_Y &\leq \psi(x, y)
\end{align*}

for all $x, y, x_j \in X$ ($1 \leq j \leq m$), where $\varphi : X^m \to [0, \infty)$ satisfies (2.1) and (2.2), and $\psi : X \times X \to [0, \infty)$ satisfies the following

\begin{equation}
\lim_{n \to \infty} \frac{1}{m^n} \psi(m^nx, m^ny) = 0
\end{equation}

for all $x, y \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H : X \to Y$ such that

\begin{equation}
\|f(x) - H(x)\|_Y \leq \frac{1}{m} \left[\tilde{\varphi}(x)\right]^\frac{1}{p}
\end{equation}

for all $x \in X$.

Proof. By Theorem 2.1, there exists a unique additive mapping $H : X \to Y$ satisfying (3.4). The additive mapping $H : X \to Y$ is given by

\begin{equation}
H(x) := \lim_{n \to \infty} \frac{1}{m^n} f(m^nx)
\end{equation}
for all $x \in X$. By the same reasoning as in the proof of Theorem of [17], the mapping $H: X \to Y$ is $\mathbb{R}$-linear.

It follows from (3.2) that
\[
\|H(xy) - H(x)H(y)\|_Y = \lim_{n \to \infty} \frac{1}{m^{2n}} \|f(m^{2n}xy) - f(m^nxf(m^ny))\|_Y \\
\leq \lim_{n \to \infty} \frac{1}{m^{2n}} \psi(m^n x, m^n y) = 0
\]
for all $x, y \in X$. Hence, we get
\[
H(xy) = H(x)H(y)
\]
for all $x, y \in X$. Thus the mapping $H: X \to Y$ is a unique homomorphism satisfying (3.4). This completes the proof of the theorem. □

**Theorem 3.2.** Suppose that a mapping $f: X \to Y$ satisfies
\[
\|Df(x_1, \cdots, x_m)\|_Y \leq \phi(x_1, \cdots, x_m)
\]
(3.6)
\[
\|f(xy) - f(x)f(y)\|_Y \leq \Psi(x, y)
\]
(3.7)
for all $x, y, x_j \in X$ ($1 \leq j \leq m$), where $\phi: X^m \to [0, \infty)$ satisfies (2.7) and (2.8), and $\Psi: X \times X \to [0, \infty)$ satisfies the following
\[
\lim_{n \to \infty} m^n \Psi\left(\frac{x}{m^n}, \frac{y}{m^n}\right) = 0
\]
(3.8)
for all $x, y \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H: X \to Y$ such that
\[
\|f(x) - H(x)\|_Y \leq \frac{1}{m} \left[\tilde{\phi}(x)\right]^\frac{1}{p}
\]
(3.9)
for all $x \in X$.

Proof. By Theorem 2.2, there exists a unique additive mapping $H : X \to Y$ satisfying (3.9). The additive mapping $H : X \to Y$ is given by

$$H(x) := \lim_{n \to \infty} m^n f\left(\frac{x}{m^n}\right)$$

for all $x \in X$. By the same reasoning as in the proof of Theorem of [17], the mapping $H : X \to Y$ is $\mathbb{R}$-linear.

It follows from (3.8) that

$$\|H(xy) - H(x)H(y)\|_Y = \lim_{n \to \infty} m^{2n}\|f\left(\frac{xy}{m^n} \cdot m^n\right) - f\left(\frac{x}{m^n}\right)f\left(\frac{y}{m^n}\right)\|_Y$$

$$\leq \lim_{n \to \infty} m^{2n}\Psi\left(\frac{x}{m^n}, \frac{y}{m^n}\right) = 0$$

for all $x, y \in X$. Hence, we get

$$H(xy) = H(x)H(y)$$

for all $x, y \in X$. Thus the mapping $H : X \to Y$ is a unique homomorphism satisfying (3.9). This completes the proof of the theorem. □

**Corollary 3.1.** Let $\theta, \delta$ be non-negative real numbers and let $r_j (1 \leq j \leq m)$, $s_1, s_2$ be non-negative real numbers such that $r_j > 1$, $s_1, s_2 > 2$ or $0 < r_j < 1$, $s_1, s_2 < 2$. Suppose that a mapping $f : X \to Y$ satisfies

$$\|Df(x_1, \cdots, x_m)\|_Y \leq \theta \sum_{i=1}^m \|x_i\|_{X_i}^{r_i}$$

(3.11)

$$\|f(xy) - f(x)f(y)\|_Y \leq \delta(\|x\|_{X}^{s_1} + \|y\|_{X}^{s_2})$$

(3.12)
for all \( x, y, x_j \in X \) (\( 1 \leq j \leq m \)). If \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \), then there exists a unique homomorphism \( H : X \to Y \) such that

\[
\| f(x) - H(x) \|_Y \leq \frac{\theta}{|m^p - m^{p\gamma}|^{\frac{1}{p}}} \| x \|_X^{\gamma}
\]

for all \( x \in X \).

**Proof.** The result follows from the proofs of Theorems 3.1 and 3.2. \( \square \)

**Corollary 3.2.** Let \( \theta, \delta \) be non-negative real numbers and let \( r_j \) (\( 1 \leq j \leq m \)), \( s_1, s_2 \) be non-negative real numbers such that \( \sum_{i=1}^{m} r_i > 1 \), \( s_1 + s_2 > 2 \) or \( \sum_{i=1}^{m} r_i < 1 \), \( s_1 + s_2 < 2 \) and \( r_j \neq 0 \) for some \( j \) (\( 2 \leq j \leq m \)). Suppose that a mapping \( f : X \to Y \) satisfies

\[
\| Df(x_1, \cdots, x_m) \|_Y \leq \theta \prod_{i=1}^{m} \| x_i \|_X^{r_i}
\]

(3.13)

\[
\| f(xy) - f(x)f(y) \|_Y \leq \delta \| x \|_X^{s_1} \| y \|_X^{s_2}
\]

(3.14)

for all \( x, y, x_j \in X \) (\( 1 \leq j \leq m \)). If \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \), then the mapping \( f : X \to Y \) is a homomorphism.

**Proof.** The result follows from the proofs of Theorems 3.1 and 3.2. \( \square \)

4. **Isomorphisms between quasi-Banach algebras**

Throughout this section, assume that \( X \) is a quasi-Banach algebra with quasi-norm \( \| \cdot \|_X \) and unit \( e \) and that \( Y \) is a \( p \)-Banach algebra with \( p \)-norm \( \| \cdot \|_Y \) and unit \( e' \). Let \( K \) be the modulus of concavity of \( \| \cdot \|_Y \).
We investigate isomorphisms between quasi-Banach algebras associated to the additive functional equation (1.1).

**Theorem 4.1.** Suppose that \( f : X \to Y \) is a bijective mapping satisfying (3.1) such that
\[
(4.1) \quad f(xy) = f(x)f(y)
\]
for all \( x, y \in X \). If \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \) and \( \lim_{n \to \infty} \frac{1}{m^n} f(m^ne) = e' \), then the mapping \( f : X \to Y \) is an isomorphism.

**Proof.** By Theorem 3.1, there exists a homomorphism \( H : X \to Y \) satisfying (3.4). The mapping \( H : X \to Y \) is given by
\[
(4.2) \quad H(x) := \lim_{n \to \infty} \frac{1}{m^n} f(m^nx)
\]
for all \( x \in X \).

By (4.1), we have
\[
H(x) = H(ex) = \lim_{n \to \infty} \frac{1}{m^n} f(m^nx) = \lim_{n \to \infty} \frac{1}{m^n} f(m^ne \cdot x) = \lim_{n \to \infty} \frac{1}{m^n} f(m^ne)f(x) = e'f(x) = f(x)
\]
for all \( x \in X \). So the bijective mapping \( f : X \to Y \) is an isomorphism. This completes the proof of the theorem. \( \square \)

**Theorem 4.2.** Suppose that \( f : X \to Y \) is a bijective mapping satisfying (3.6) and (4.1). If \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \) and \( \lim_{n \to \infty} m^n f(\frac{e}{m^n}) = e' \), then the mapping \( f : X \to Y \) is an isomorphism.
Proof. By Theorem 3.2, there exists a homomorphism \( H: X \to Y \) satisfying (3.9). The mapping \( H: X \to Y \) is given by

\[
H(x) := \lim_{n \to \infty} m^n f \left( \frac{x}{m^n} \right)
\]

for all \( x \in X \).

The rest of the proof is similar to the proof of Theorem 4.1. This completes the proof of the theorem. \qed

Corollary 4.1. Let \( \theta, r_j \) (\( 1 \leq j \leq m \)) be non-negative real numbers such that \( r_j > 1 \) or \( 0 < r_j < 1 \). Suppose that a bijective mapping \( f: X \to Y \) satisfies (3.11) and (4.1). If \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \) and \( \lim_{n \to \infty} m^n f \left( \frac{e}{m^n} \right) = e' \) or \( \lim_{n \to \infty} \frac{1}{m^n} f(m^n e) = e' \), then the mapping \( f: X \to Y \) is an isomorphism.

Proof. The result follows from the proofs of Theorems 4.1 and 4.2. \qed

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