ONE PROPERTY OF THE WEAK COVERAGE OF OPERATORS ITERATIONS IN VON NEUMANN ALGEBRAS

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Conditions are given for *-weak convergence of iterations for an ultraweak continuous functional in von Neumann algebra to imply norm convergence.

Let $M$ be a von Neumann algebra [5], acting on a separable Hilbert space $H$. Let $T$ be a contraction from $M_+$ to $M_*$, so that $TM_+ \subseteq M_+$. On the pre-conjugate to $M$ space $M_*$ there are two topologies selected: the weak, or the $\sigma(M_*, M)$ topology, and the strong topology of the convergence in the norm of the space $M_*$.

Let now $T = \alpha$, where $\alpha$ be an automorphism of the algebra $M$. We will say that $T$ in $M_*$ is mixing, if for all $x \in M_*$ and $A \in M$, the following condition is valid:

$$\lim_{n \to \infty} \langle T^n x, A \rangle = 0,$$

where

$$M_*^0 = \{ y \in M_* : y(1) = 0 \}.$$

We will say that a positive contraction $T$ in $M_*$ is completely mixing, if for all $x \in M_*^0$ the following condition is valid:

$$\lim_{n \to \infty} \| T^n x \| = 0.$$  

The following theorem is valid:

**Theorem.** Let $T$ be a pre-conjugate operator to an automorphism $\alpha$ of a von Neumann algebra $M$ for which there is no invariant normal state. Then, for $x \in M_*$, the weak convergence of $T^n x$ implies the strong convergence of $T^n x$. In particular, if $T$ is mixing, then $T$ is completely mixing.

< Let us denote by $|T^n x|$ the sum

$$(T^n x)_+ + (T^n x)_-,$$

where $T^n x = (T^n x)_+ - (T^n x)_-$

is the Hahn decomposition of the functional $T^n x$ [4]. The sequence $\{ |T^n x| \}_{n=1}^\infty$ is $\sigma(M_*, M)$ pre-compact [4] and, therefore, the convex envelope of the set $\{ |T^n x| \}_{n=1}^\infty$ is pre-compact as well. The sequence $\{ A^n |x| \}_{n=1}^\infty$ is also pre-compact because it belongs to the convex envelope of the set $\{ |T^n x| \}_{n=1}^\infty$.

Because $T$ is pre-conjugate to an automorphism, then $|T^n x| = T^n |x|$. In fact, the support of $T(T^n x)_+$ is orthogonal to the support of $T(T^n x)_-$. $T(T^n x)_+ - T(T^n x)_- = T(T^n x) = T^{n+1} x$, and from the uniqueness of the Hahn decomposition [4] it follows that $|T^n x| = T^n |x|$.

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Let $\pi$ be $\sigma(M_e, M)$-limit point of the set $\{A^n \|x\|\}_{n=1}^{\infty}$. Then the functional $\pi$ will be $T$-invariant. In fact,

$$T\pi = \lim_{n \to \infty} \sum_{k=1}^{n} \left< T^k x, y \right>$$

$$= \lim_{n \to \infty} \left[ n^{-1} \cdot \sum_{k=0}^{n-1} \left< T^k x, y \right> - n^{-1} \cdot \left< x, y \right> + n^{-1} \cdot \left< T^n x, y \right> \right] = \pi.$$

It is easy to see that $\pi \geq 0$ and, therefore, from the conditions of the theorem it follows that $\pi = 0$. Now we know that the only weakly limit point of the set $\{A^n \|x\|\}_{n=1}^{\infty}$ is the point $\pi = 0$. Therefore

$$0 = \lim_{n \to \infty} \|A^n \|x\|\| = \lim_{n \to \infty} (A^n \|x\|)(1) = \lim_{n \to \infty} (T^n \|x\|)(1) = \lim_{n \to \infty} ||T^n \|x\||,$$

because $(T^n \|x\|)(1) = (T^m \|x\|)(1)$ for all $n, m \in \mathbb{N}$. The theorem is proven. $\triangleright$

References


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